University of Canterbury

Masters Thesis

Embeddable Spherical Circle Planes

Author:
Ashley Lightfoot

Supervisor:
Dr. Günter Steinke

September, 2009

A thesis submitted in partial fulfilment of the requirements of the Degree for Master of Science in Mathematics

UC
UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

University of Canterbury, Private Bag 4800, Christchurch 8140, New Zealand

Abstract

Spherical circle planes are topological incidence geometries; one has a 2-sphere $\mathcal{P}$ and a collection of 1-spheres in $\mathcal{P}$ such that any three points in $\mathcal{P}$ determine exactly one of these 1-spheres (the 'circles' of the spherical circle plane). A determination of the homeomorphism type of the circle space (the collection of 1-spheres suitably topologised) of a topological 2-dimensional Möbius plane, which is a spherical circle plane with the additional property of the axiom of touching, was given by Karl Strambach in 1974. Embeddable spherical circle planes are a type of spherical circle plane that are not, in general, Möbius planes, constructed on a 2-sphere $\mathbf{P}$ in $\mathbb{R}^3$ by taking the circles to be precisely the non-trivial plane intersections in $\mathbb{R}^3$ with $\mathbf{P}$. We show that the circle space of an embeddable spherical circle plane is homeomorphic to the 3-dimensional projective space minus one point. Furthermore, it is shown that the flag space of an embeddable spherical circle plane is homeomorphic to the flag space of the classical flat Möbius plane; a topological description of the latter is also given.

An apparent gap is the literature is also filled: we prove the well-known conjecture that the flag space of a general spherical circle plane is a 4-dimensional manifold.

Finally, we define the notion of isotopy equivalence between spherical circle planes and prove that embeddable spherical circle planes are isotopy equivalent to the classical flat Möbius plane.

Contents

Chapter 1

Introduction

1.1 Motivation and Thesis Outline

Möbius planes have their origins in the work of A.F. Möbius in the 19th century and his investigations of properties of Euclidean lines and Euclidean circles. Möbius planes are incidence geometries consisting of two kinds of objects, points and circles. In the same way an affine plane axiomatises the geometry of Euclidean lines in the Euclidean plane, Möbius planes axiomatise the geometry obtained from an elliptic quadric in 3-dimensional projective space and all its (non-trivial) intersections with planes in 3-space.

Möbius planes, or inversive planes as they are also called, have been investigated as abstract geometries or with additional assumptions like finitely many points (this leads to certain 3-designs) or topological conditions. D. Wölk [13] introduced topological Möbius planes, where the set of points and the set of circles carry topologies and continuity conditions on the geometric operations are imposed. Simple and easy-to-visualise models can be obtained by replacing an ellipsoid in real 3-space by a surface that is the boundary of a strictly convex compact body and again considering its intersections with planes. By dropping smoothness conditions on the surface one obtains a further generalisation, the embeddable spherical circles planes, which are the main object of study in this thesis.

The emphasis here is on exploring the topological aspects of such geometries; we spend more time in Euclidean space than we do in arbitrary geometries of lines and points.

The aim of this thesis is to extend the current knowledge of spherical circle planes by two means. The first is to investigate what is known only about the topological structures associated with the subclass of flat Möbius planes, and seek to expand this family to include embeddable spherical circle planes. The second is to transpose the notion of isotopy equivalence, as defined with respect to linear spaces, onto spherical circle planes; in doing so we introduce a new measure for comparing spherical circle planes.

Strambach [11] classified the circle space of a flat Möbius plane — namely, that the circle space is homeomorphic to the 3-dimensional projective space minus one point — but his proof relied on the additional property, called the axiom of touching, which these planes possess.

Embeddable spherical planes are spherical circle planes that are not, in general, flat Möbius planes. Motivated by Strambach's work in 1970 which showed that spherical circle planes are topological circle planes, in Chapter 3 we prove directly that this is indeed the case for embeddable spherical circle planes. This gives a justification for the topology chosen on the circle set of an embeddable spherical circle plane.

During the work in Chapter 3, we formulate some of the tools necessary to prove, in Chapter 4, the new result that the circle space of an embeddable spherical circle plane is homeomorphic to the 3-dimensional projective space minus one point. This investigation leads naturally to the task of classifying the flag space of an embeddable spherical circle plane. In the latter part of Chapter 5, in another new result, we construct an explicit homeomorphism from the flag space of an embeddable spherical circle plane to that of the classical flat Möbius plane; a topological description of the latter is then given. Curiously, a gap in the literature was discovered during this process: Polster and Steinke [6] state without any reference to a proof that the flag space of a general spherical circle plane is a 4-dimensional manifold. So, to begin Chapter 5, we make a detour beyond the realm of embeddable spherical circle planes to fill this hole and prove the conjecture.

The techniques arising in the pursuit of constructing a homeomorphism between the flag spaces of an embeddable spherical circle plane and the classical flat Möbius plane lead to a connection between spherical circle planes and Rosehr's [7] formulation of an isotopy between stable planes. In Chapter 6, in our final new result, we provide an analogous definition of an isotopy between spherical circle planes and show that embeddable spherical circle planes are, according to this definition, isotopy equivalent to the classical flat Möbius plane.

Because of the quantity of different maps and spaces introduced in the thesis, we have included a list of symbols for the reader's convenience.

1.2 Acknowledgements

It is with great pleasure that I acknowledge my supervisor, Dr. Günter Steinke, whose guidance has been the backbone of this thesis. I would feel immensely privileged if only a small fraction of his insight and creativity in formulating conjectures, and ability to remain optimistic when one attack on a problem fails, rubs off on me. Not only has Dr. Steinke supervised this thesis, but he

has also supervised an Honours project and two summer projects of mine in the last three years during my time as an undergraduate student at the University of Canterbury. I am grateful, therefore, to have been able to rely on his mentorship to help instill in me the skills of a research mathematician; it is only fair that I now let new undergraduate students take advantage of his deep intellect and voluminous generosity.

Chapter 2

Preliminaries

2.1 Preliminary Definitions

We begin by defining some important terminology that will arise in this thesis. Most of the nomenclature is the same as that of [6], albeit slightly narrower to minimally accommodate our specific considerations.

In this thesis a *geometry* will consist of a nonempty set called the *point set* and either a set of *lines* or a set of *circles*. A line (circle) is a subset of the point set containing at least three points[1] that is uniquely determined by two (three) distinct points.

Two or more points are said to be *collinear* (*cocircular*) if they are contained in some line (circle); we often diverge from the strict language of sets and say, for example, that a point *lies on* a circle, rather than is *contained in* a circle. A *point-line geometry* is a pair $(\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$ is the *point-set* and $\mathcal{L}$ is the set of *lines*, such that any two distinct points lie on a uniquely determined line.

A *point-circle geometry* is a pair $(\mathcal{P}, \mathcal{C})$, where $\mathcal{P}$ is the point-set and $\mathcal{C}$ is the

[1] In this thesis, the term "point" shall, as usual, be a synonym for "element" in the context of an arbitrary set and is not limited to elements of the point set of a geometry.

set of *circles*, such that any three distinct points lie on a uniquely determined circle. Furthermore, each circle contains at least three points and there are at least two circles. When $\mathcal{P}$ and $\mathcal{C}$ have been suitably topologised, we refer to the resulting spaces as the *point space* and *circle space*, respectively. We let $\widetilde{\mathcal{P}^2}$ denote the quotient space of the product space $\mathcal{P}^2$ under the equivalence relation

$$(\mathbf{x},\mathbf{y}) \sim (\mathbf{u},\mathbf{v}) \Leftrightarrow \big\{(\mathbf{x},\mathbf{y}) = (\mathbf{u},\mathbf{v}) \text{ or } (\mathbf{x},\mathbf{y}) = (\mathbf{v},\mathbf{u})\big\}.$$

If $\mathbf{p} \in \mathcal{P}$ and $C \in \mathcal{C}$ with $\mathbf{p} \in C$, we call the ordered pair $(\mathbf{p}, C)$ a *flag*. We endow the set of flags

$$\mathcal{F} := \{(\mathbf{p}, C) \subseteq \mathcal{P} \times \mathcal{C} : \mathbf{p} \in C\}$$

with the subspace topology with respect to $\mathcal{P} \times \mathcal{C}$, and call the resulting space the *flag space* $\mathcal{F}$ of $(\mathcal{P}, \mathcal{C})$.

A point-circle geometry $(\mathcal{P}, \mathcal{C})$ is called a *Möbius plane* if for each circle $C \in \mathcal{C}$ and any two distinct points $\mathbf{p}$, $\mathbf{q} \in \mathcal{P}$ with $\mathbf{p} \in \mathcal{C}$ and $\mathbf{q} \notin \mathcal{C}$, there is precisely one circle C' containing $\mathbf{p}$ and $\mathbf{q}$ such that $C \cap C' = \mathbf{p}$. This additional property of Möbius planes is called the *axiom of touching*.

2.2 Topological Conventions

A topological space, that is, a set equipped with a topology[2], shall be referred to simply as a *space*. If A is a subset of a space X, then $\overline{A}$ denotes its closure; we denote its boundary $\overline{A} \backslash A^\circ$ by ∂A. We write $A^c := X \backslash A$; if Y is a space homeomorphic to X, we may write $X \approx Y$. We denote the identity mapping $x \mapsto x$ on X by id_X .

A *neighbourhood* of an element $x \in X$ is a subset of X that contains an open

[2] See [5] for a definition.

subset B such that $x \in B$. A *neighbourhood basis* for an element $x \in X$ is a collection $\mathcal{N}$ of neighbourhoods of x such that for any neighbourhood U of x, there is a neighbourhood $B \in \mathcal{N}$ such that $B \subseteq U$. An *m-dimensional manifold* is a Hausdorff space X with a countable basis (that is, X is second countable) such that each element $x \in X$ has a neighbourhood that is homeomorphic to an open subset of $\mathbb{R}^m$.

In a metric space[3] (X, ρ), denote the open ball of radius $r > 0$ with centre $\mathbf{x}$ by

$$\mathcal{B}_\rho(\mathbf{x}, r) := \{\mathbf{y} \in X : \rho(\mathbf{x}, \mathbf{y}) < r\}.$$

If the choice of metric ρ is clear we abbreviate this $\mathcal{B}(\mathbf{x}, r) := \mathcal{B}_\rho(\mathbf{x}, r)$. Let $d \in \mathbb{N}$. The Euclidean metric ρ_d on $\mathbb{R}^d$ is that induced by the Euclidean norm

$$||(x_1, x_2, \ldots, x_d)|| := \left(\sum_{i=1}^{d} |x_i|\right)^{\frac{1}{d}};$$

that is,

$$\rho_d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||.$$

The *d-sphere* $\mathbb{S}_r^d$ of radius r is the boundary of $\mathcal{B}_{\rho_{d+1}}(\mathbf{0}, r)$, that is,

$$\mathbb{S}_r^d = \{\mathbf{x} \in \mathbb{R}^d : ||\mathbf{x}|| = r\}.$$

We shall frequently move between the Euclidean space $\mathbb{R}^{d+1}$ and the *d-dimensional projective space* $\mathbb{P}_d\mathbb{R}$. Denote the linear span of a vector $\mathbf{x} \in \mathbb{R}^{d+1} \backslash \mathbf{0}$[4,5] by

$$[\mathbf{x}] := \{k\mathbf{x} : k \in \mathbb{R}^{d+1}\}.$$

[3]See [5] for a definition.
[4]When the value of d is clear we denote the zero vector in $\mathbb{R}^d$ by $\mathbf{0}$.
[5]We adopt the abbreviation $\mathbf{a}$ for a singleton $\{\mathbf{a}\}$.

Whence

$$\mathbb{P}_d\mathbb{R} := \left\{[\mathbf{x}] : \mathbf{x} \in \mathbb{R}^{d+1}\backslash\mathbf{0}\right\};$$

we equip $\mathbb{P}_d\mathbb{R}$ with the quotient topology on $\mathbb{R}^{d+1}\backslash\mathbf{0}$ defined by the map

$$\pi_d : \mathbb{R}^{d+1}\backslash\mathbf{0} \to \mathbb{P}_d\mathbb{R} : \mathbf{x} \mapsto [\mathbf{x}].$$

This topology is precisely the topology induced by the metric

$$\varrho_d : \mathbb{P}_d\mathbb{R} \times \mathbb{P}_d\mathbb{R} \to \mathbb{R}_{\geq 0}$$

given by

$$\varrho_d([\mathbf{x}], [\mathbf{y}]) = \min\left\{\left\|\frac{\mathbf{x}}{||\mathbf{x}||} + \frac{\mathbf{y}}{||\mathbf{y}||}\right\|, \left\|\frac{\mathbf{x}}{||\mathbf{x}||} - \frac{\mathbf{y}}{||\mathbf{y}||}\right\|\right\},$$

for $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^{d+1}\backslash\mathbf{0}$ (cf. [6], pp. 430).

A sequence[6] in $\mathbb{P}_d\mathbb{R}$ is of the form $([\mathbf{x}_n])$, where $(\mathbf{x}_n)$ is a sequence in $\mathbb{R}^{d+1}\backslash\mathbf{0}$; convergence in $\mathbb{P}_d\mathbb{R}$ is given by:

$$[\mathbf{x}_n] \to [\mathbf{x}] \text{ if and only if } \varrho_d([\mathbf{x}_n], [\mathbf{x}]) \to 0.$$

Let $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$ and put $\mathbf{u}_n = \frac{\mathbf{x}_n}{||\mathbf{x}_n||}$ for each n. Note that if $\mathbf{u}_n \to \mathbf{u}$ in $\mathbb{R}^{d+1}\backslash\mathbf{0}$, then

$$\min\left\{\|\mathbf{u}_n + \mathbf{u}\|, \|\mathbf{u}_n - \mathbf{u}\|\right\} \leq ||\mathbf{u}_n - \mathbf{u}|| \to 0,$$

so $[\mathbf{x}_n] \to [\mathbf{x}]$.

Proposition 2.2.1. *Let $([\mathbf{x}_n])$ be a sequence in $\mathbb{P}_d\mathbb{R}$ (for $d \in \mathbb{N}$) and let $\mathbf{x} \in \mathbb{R}^{d+1}\backslash\mathbf{0}$. Let $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$ and put $\mathbf{u}_n = \frac{\mathbf{x}_n}{||\mathbf{x}_n||}$ for each n; suppose that $(\mathbf{u}_n)$ does not converge to $\mathbf{u}$ in $\mathbb{R}^{d+1}$. Then there exist subsequences $(\mathbf{u}_m)_{m\in N_1}$, $(\mathbf{u}_{m'})_{m'\in N_1}$*

[6] We adopt the abbreviation (y_n) for a sequence $(y_n)_{n\in\mathbb{N}}$, and write $y_n \to y$ as short-hand for $y_n \xrightarrow{n\to\infty} y$.

of $(\mathbf{u}_n)$ *corresponding to the elements for which*

$$\|\mathbf{u}_m - \mathbf{u}\| = \varrho_d([\mathbf{x}_m], [\mathbf{x}])$$

and

$$\|\mathbf{u}'_m + \mathbf{u}\| = \varrho_d([\mathbf{x}'_m], [\mathbf{x}]),$$

respectively, and where N_1 *and* N_2 *partition* $\mathbb{N}$*. Furthermore,* $[\mathbf{x}_n] \to [\mathbf{x}]$ *if and only if*

$$\mathbf{u}_m \to \mathbf{u} \xrightarrow{m\to\infty} \mathbf{u} \;\; \textit{and} \;\; \mathbf{u}_{m'} \xrightarrow{m'\to\infty} -\mathbf{u}$$

in $\mathbb{R}^{d+1}$*.*

Proof. If $[\mathbf{x}_n] \to [\mathbf{x}]$, then we have

$$\|\mathbf{u}_m - \mathbf{u}\| = \varrho_d([\mathbf{x}_m], [\mathbf{x}]) \xrightarrow{m\to\infty} 0$$

and

$$\|\mathbf{u}_{m'} + \mathbf{u}\| = \varrho_d([\mathbf{x}_{m'}], [\mathbf{x}]) \xrightarrow{m'\to\infty} 0.$$

That is, $\mathbf{u}_m \xrightarrow{m\to\infty} \mathbf{u}$ and $\mathbf{u}_{m'} \xrightarrow{m'\to\infty} -\mathbf{u}$.

Conversely, suppose that $\mathbf{u}_m \xrightarrow{m\to\infty} \mathbf{u}$ and $\mathbf{u}_{m'} \xrightarrow{l\to\infty} -\mathbf{u}$; let $\varepsilon > 0$. There are $K, L > 0$ such that

$$||\mathbf{v}_m - \mathbf{u}|| < \varepsilon \text{ and } ||\mathbf{w}'_m + \mathbf{u}|| < \varepsilon$$

for all $m > K$ and $m' > L$, respectively. For each $n > \max\{K, L\}$, we therefore have that

$$\varrho_d([\mathbf{x}_n],[\mathbf{x}]) = \begin{cases} ||\mathbf{u}_m - \mathbf{u}|| & \text{if } n = m \in N_1, \\ ||\mathbf{u}_{m'} + \mathbf{u}|| & \text{if } n = m' \in N_2 \end{cases}$$
$$< \varepsilon.$$

This shows that $[\mathbf{x}_n] \to [\mathbf{x}]$. $\square$

Let $\mathbf{p}$, $\mathbf{q}$ be distinct points in $\mathbb{R}^d$. Denote the open line segment between them by

$$]\mathbf{p},\mathbf{q}[:= \{t\mathbf{p} + (1-t)\mathbf{q} : t \in (0,1)\};$$

denote the open ray emanating from the *basepoint* $\mathbf{x} \in \mathbb{R}^d$ in the *direction* $\mathbf{u} \in \mathbb{S}_1^2$ by

$$\mathcal{R}(\mathbf{x},\mathbf{u}) := \{\mathbf{x} + k\mathbf{u} : k \in \mathbb{R}_{>0}{}^{[7]}\},$$

and the line[8] with a (non-unique) basepoint $\mathbf{x} \in \mathbb{R}^d$ and direction $[\mathbf{u}] \in \mathbb{P}_2\mathbb{R}$ by

$$L(\mathbf{x},[\mathbf{u}]) := \{\mathbf{x} + k\mathbf{u} : k \in \mathbb{R}\}.$$

The closed analogues of these three objects are defined by taking the respective closures. A *convex* set is a closed subset A of $\mathbb{R}^d$ (for some $d \in \mathbb{N}$) with the property that whenever $\mathbf{p},\mathbf{q} \in A$, the open (equivalently, closed) line segment between $\mathbf{p}$ and $\mathbf{q}$ is contained in A.

A plane E separates $\mathbb{R}^3$, that is, $\mathbb{R}^3\backslash E$ is not connected and has the connected components H_1 and H_2. We call H_1 and H_2 ($\overline{H_1}$ and $\overline{H_2}$) the open (closed) *half-spaces* determined by E.

[7]Denote $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$, and $\mathbb{R}_{\geq 0}, \mathbb{R}_{<0}, \mathbb{R}_{\leq 0}$ analogously.
[8]Outside of the context of a prescribed point-line geometry, we use the term “line” in the Euclidean sense.

2.3 Embeddable Spherical Circle Planes

A *spherical circle plane*, abbreviated *SCP*, is a point-circle geometry whose point set is homeomorphic to $\mathbb{S}_1^2$ and whose circles, as subsets of the point space, are homeomorphic to $\mathbb{S}_1^1$.

We proceed to define the point-circle geometry $(\mathbf{P}, \mathbf{C})$ called an *embeddable spherical circle plane*. The point space $\mathbf{P}$ is a subset of $\mathbb{R}^3$ homeomorphic to $\mathbb{S}_1^2$. Furthermore, $\mathbf{P}$ satisfies a "strict convexity" condition which we now describe. By the Jordan-Brouwer separation theorem (Theorem A.1.13), $\mathbb{R}^3 \backslash \mathbf{P}$ has two connected components: a bounded component $\mathbf{B}$, and an unbounded component. We impose the condition that the space $\overline{\mathbf{B}} = \mathbf{P} \cup \mathbf{B}$ is convex (as a subset of $\mathbb{R}^3$) and also that any open line segment between two points in $\overline{\mathbf{B}}$ is contained in $\mathbf{B}$. More generally, we say that a subset A of $\mathbb{R}^3$ is *strictly convex* if any line in $\mathbb{R}^3$ intersects A in at most two points. Later it proves useful for coordinatizing purposes to make the assumption that $\mathbf{0} \in \mathbf{B}$.

We define the circles of $\mathbf{C}$ to be precisely the intersections of planes in $\mathbb{R}^3$ with $\mathbf{P}$ containing at least three points. Notice that we may then unambiguously refer to $(\mathbf{P}, \mathbf{C})$ as *the* embeddable spherical circle plane with point set $\mathbf{P}$.

For $(a, b, c) \neq (0, 0, 0)$, we adopt the notation

$$E(a, b, c, d) := \{(x_1, x_2, x_3) : ax_1 + bx_2 + cx_3 = d\}$$

for a plane in $\mathbb{R}^3$. Since $E(a, b, c, d) = E(a', b', c', d')$ if and only if $[(a, b, c, d)] = [(a', b', c', d')]$, with (a, b, c), $(a', b', c') \neq (0, 0, 0)$, it is natural to identify planes in $\mathbb{R}^3$ with precisely the points of $\mathbb{P}_3\mathbb{R} \backslash \mathbf{p_o}$, where $\mathbf{p}_o := [(0, 0, 0, 1)]$. The topology on the set of planes is then identified with the subspace topology of $\mathbb{P}_3\mathbb{R}$ accordingly.

Let us now confirm that $(\mathbf{P}, \mathbf{C})$ is a point-circle geometry by verifying that three distinct points of $\mathbf{P}$ uniquely determine a circle $C \in \mathbf{C}$.

Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be mutually distinct points of $\mathbf{P}$. If $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are collinear, then the line in $\mathbb{R}^3$ through them intersects $\mathbf{P}$ at three points — contradicting the strict convexity of $\mathbf{P}$. Thus $\mathbf{x} - \mathbf{y}$ and $\mathbf{x} - \mathbf{z}$ are linearly independent, so $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are contained uniquely in the plane $E(a, b, c, d)$ and hence the circle $\mathbf{P} \cap E(a, b, c, d)$, where [9]

$$(a, b, c) = (\mathbf{x} - \mathbf{y}) \boxtimes (\mathbf{x} - \mathbf{z}) \quad \text{and}$$
$$d = ax_1 + bx_2 + cx_3,$$

where $\boxtimes$ denotes the cross-product in $\mathbb{R}^3$, .

2.3.1 ESCPs are SCPs

We now prove directly that an embeddable spherical circle plane is indeed a spherical circle plane. As the point set $\mathbf{P}$ is homeomorphic to $\mathbb{S}^2_1$ by definition, we are required to show that each circle of $\mathbf{C}$ is homeomorphic to $\mathbb{S}^1_1$.

Proposition 2.3.1. *For each $\mathbf{u} \in \mathbb{S}^2_1$, the open ray $\mathcal{R}(\mathbf{0}, \mathbf{u})$ intersects $\mathbf{P}$ at exactly one point.*

Proof. First suppose that $\mathcal{R}(\mathbf{0}, \mathbf{u}) \cap \mathbf{P} = \varnothing$. Since $\mathbf{0} \in \mathbf{B}$ and $\mathbf{B}$ is open, there is an $r > 0$ such that the open ball $\mathcal{B}(\mathbf{0}, r)$ is contained in $\mathbf{B}$. The connected ray $\mathcal{R}(\mathbf{0}, \mathbf{u})$ has non-empty intersection with $\mathcal{B}(\mathbf{0}, r)$ and hence with $\mathbf{B}$, so $\mathcal{R}(\mathbf{0}, \mathbf{u}) \subseteq \mathbf{B}$. But $\mathcal{R}(\mathbf{0}, \mathbf{u})$ is unbounded, and $\mathbf{B}$ is bounded — a contradiction. Thus $\mathcal{R}(\mathbf{0}, \mathbf{u})$ intersects $\mathbf{P}$.

[9] If $\mathbf{u}$ lies on the plane through $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, then using the point normal form of a plane, $\mathbf{n} \cdot (\mathbf{u} - \mathbf{u}_0) = 0$, with $\mathbf{u}_0 = \mathbf{x}$ and $\mathbf{n} = (\mathbf{x} - \mathbf{y}) \boxtimes (\mathbf{x} - \mathbf{z})$, we have

$$\begin{aligned} 0 &= \big((\mathbf{x} - \mathbf{y}) \boxtimes (\mathbf{x} - \mathbf{z})\big) \cdot (\mathbf{u} - \mathbf{x}) \\ &= \big((\mathbf{x} - \mathbf{y}) \boxtimes (\mathbf{x} - \mathbf{z})\big) \cdot \mathbf{u} - \big((\mathbf{x} - \mathbf{y}) \boxtimes (\mathbf{x} - \mathbf{z})\big) \cdot \mathbf{x} \\ &= (a, b, c) \cdot \mathbf{u} - d, \end{aligned}$$

where $(a, b, c) := (\mathbf{x} - \mathbf{y}) \boxtimes (\mathbf{x} - \mathbf{z})$ and $d = (a, b, c) \cdot \mathbf{x}$.

Now, suppose that $\mathbf{p}$ and $\mathbf{q}$ are distinct points in $\mathcal{R}(\mathbf{0},\mathbf{u}) \cap \mathbf{P}$ with $||\mathbf{p}|| < ||\mathbf{q}||$. Then $\mathbf{p}$ lies in the open line segment $]\mathbf{0},\mathbf{q}[$, which lies in $\mathbf{B}$ by the strict convexity of $\mathbf{P}$ — contradicting that $\mathbf{p} \in \mathbf{P} = \overline{\mathbf{B}} \backslash \mathbf{B}$. Hence $\mathcal{R}(\mathbf{0},\mathbf{u}) \cap \mathbf{P}$ is a single point. □

Proposition 2.3.2. *A plane E in $\mathbb{R}^3$ passing through the origin intersects $\mathbf{P}$ at more than one point.*

Proof. Let $\mathbf{y} \in E \backslash \{\mathbf{0}\}$; put $\mathbf{u} = \frac{\mathbf{y}}{||\mathbf{y}||}$. By Proposition 2.3.1, each of the open rays $\mathcal{R}(\mathbf{0},\mathbf{u})$ and $\mathcal{R}(\mathbf{0},-\mathbf{u})$, which are contained in E, intersect $\mathbf{P}$ at a unique point. Furthermore, since $\mathcal{R}(\mathbf{0},\mathbf{u})$ and $\mathcal{R}(\mathbf{0},-\mathbf{u})$ do not intersect, we obtain two distinct points of intersection $\mathcal{R}(\mathbf{0},\mathbf{u}) \cap \mathbf{P}$ and $\mathcal{R}(\mathbf{0},-\mathbf{u}) \cap \mathbf{P}$. □

Proposition 2.3.3. *A plane E in $\mathbb{R}^3$ intersects $\mathbf{P}$ at more than one point (that is,* non-trivially*) if and only if E intersects $\mathbf{B}$.*

Proof. Firstly, if E intersects $\mathbf{B}$ at a point $\mathbf{p}$, then by choosing $\mathbf{p}$ to be the origin we have from Proposition 2.3.2 that E intersects $\mathbf{P}$ non-trivially. Conversely, suppose that $\mathbf{p}$ and $\mathbf{q}$ are distinct points in $E \cap \mathbf{P}$. Then the open line segment $]\mathbf{p},\mathbf{q}[$ lies in $E \cap \mathbf{B}$ by the strict convexity of $\mathbf{P}$ and of E. □

Theorem 2.3.4. *Each circle C in the circle space $\mathbf{C}$ of an embeddable spherical circle plane is homeomorphic to the unit circle $\mathbb{S}^1_1$.*

Proof. Let $C \in \mathbf{C}$, so $C = E \cap \mathbf{P}$ for some plane E intersecting $\mathbf{P}$ non-trivially. By Proposition 2.3.3, we have that $E \cap \mathbf{B} \neq \varnothing$, so we may choose coordinates so that E is the xy plane, that is, $E = \{(x,y,0) : x,y \in \mathbb{R}\}$. In particular, the origin $\mathbf{0}$ lies in $E \cap \mathbf{B}$.

Now, define the map $h : C \to \mathbb{S}^1_1 \times \mathbf{0}$ by

$$h : \mathbf{x} \mapsto \frac{\mathbf{x}}{||\mathbf{x}||}.$$

By Proposition 2.3.1, the open ray $\mathcal{R}(\mathbf{0}, \mathbf{u})$ intersects $C = E \cap \mathbf{P}$ at exactly one point; let $j : \mathbb{S}_1^1 \times \mathbf{0} \to C$ be the mapping

$$j : \mathbf{u} \times \mathbf{0} \mapsto \mathcal{R}(\mathbf{0}, \mathbf{u}) \cap C.$$

For $\mathbf{x} \in C$ and putting $\frac{\mathbf{x}}{||\mathbf{x}||} = \mathbf{u} \times \mathbf{0}$, we have

$$\mathbf{x} \stackrel{h}{\mapsto} \mathbf{u} \stackrel{j}{\mapsto} \mathcal{R}(\mathbf{0}, \mathbf{u}) \cap C = \mathbf{x}.$$

For $\mathbf{u} \in \mathbb{S}_1^1$,

$$\mathbf{u} \times \mathbf{0} \stackrel{j}{\mapsto} \mathcal{R}(\mathbf{0}, \mathbf{u}) \cap C \stackrel{h}{\mapsto} \frac{\mathcal{R}(\mathbf{0}, \mathbf{u}) \cap C}{||\mathcal{R}(\mathbf{0}, \mathbf{u}) \cap C||} = \mathbf{u} \times \mathbf{0}.$$

This shows that $j = h^{-1}$, so h is a bijection. Since $C = E \cap \mathbf{P}$ is a closed subset of the compact space $\mathbf{P}$, it is compact. As $\mathbb{S}_1^1$ is Hausdorff, we obtain from Theorem A.1.9 that h is a homeomorphism. $\square$

This completes our direct proof that an embeddable spherical circle plane is a spherical circle plane.

Chapter 3

ESCPs are Topological Circle Planes

In this chapter we define the topology on the circle set of an embeddable spherical circle plane and justify this choice. It is a fundamental notion in the field of topological geometry that the topologies on the point and circle (or line) sets should induce certain continuity properties on certain maps. Having decided that the point space $\mathbf{P}$ is a subspace of $\mathbb{R}^3$, we wish to show that our choice of topology on $\mathbf{C}$ results in $(\mathbf{P}, \mathbf{C})$ satisfying these conditions, so in a sense is the "correct" choice.

3.1 Topological Circle Planes

Associated with a point-circle geometry are two geometric operations: joining three points by a circle, and intersecting two circles. Using the point and circle sets, we can describe these operations as maps; upon toplogising the point and circle sets, we can discuss the continuity of such maps.

For a point-circle geometry $(\mathcal{P}, \mathcal{C})$, we denote

$$(\mathcal{P}^3)_* := \big\{(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathcal{P}^3 : \ \mathbf{x},\ \mathbf{y} \text{ and } \mathbf{z} \text{ are mutually distinct}\big\},$$

and

$$(\mathcal{C}^2)_* := \{(C, D) \in \mathcal{C}^2 : \ C \text{ and } D \text{ intersect at precisely two points}\}.$$

We can then define the *joining map*

$$\alpha : (\mathcal{P}^3)_* \to \mathcal{C}$$

to be such that $\alpha(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is the unique circle determined by $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in (\mathcal{P}^3)_*$. The *intersection map*

$$\gamma : (\mathcal{C}^2)_* \to \widetilde{P^2}$$

is defined by

$$\gamma(C, D) = C \cap D.$$

With $\mathcal{P}$ and $\mathcal{C}$ suitably topologised, we say that the intersection map is *stable* if, in addition to γ being continuous, the set $(\mathcal{C}^2)_*$ is an open subset of $\mathcal{C}^2$. We call a point-circle geometry $(\mathcal{P}, \mathcal{C})$ a *topological circle plane* if the point and circle sets carry Hausdorff topologies, the joining map is continuous and the intersection map is stable.

The following result is due to Strambach ([10], Corollary 2.8).

Theorem 3.1.1. *Let $(\mathcal{P}, \mathcal{C})$ be a point-circle geometry, where $\mathcal{P}$ is a connected and 2-dimensional topological space. Then $(\mathcal{P}, \mathcal{C})$ is a spherical circle plane if and only if it is a topological circle plane.*

We verify that the forward direction of Theorem 3.1.1 does indeed hold for the case of embeddable spherical circle planes. Let $(\mathbf{P}, \mathbf{C})$ be an embeddable spherical circle plane; then $\mathbf{P} \approx \mathbb{S}_1^2$ is Hausdorff, connected and 2-dimensional. We show that the circle space is Hausdorff, the joining map is continuous and the intersection map is stable.

3.2 The Topology on the Circle Set

Recall that the set of planes in $\mathbb{R}^3$ is endowed with the topology identifying them with the subspace $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$ of $\mathbb{P}_3\mathbb{R}$, where $\mathbf{p}_o = [(0,0,0,1)]$: a plane $E(a,b,c,d)$, with $(a,b,c) \neq (0,0,0)$, is identified with the element $[(a,b,c,d)]$ of $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$.

Since each $C \in \mathbf{C}$ contains three non-collinear intersection points of a plane in $\mathbb{R}^3$ and $\mathbf{P}$, there is a bijection between $\mathbf{C}$ and the set of planes in $\mathbb{R}^3$ which intersect $\mathbf{P}$ in at least three points. We choose the topology on $\mathbf{C}$ so that this bijection is a homeomorphism, and henceforth regard $\mathbf{C}$ to be a subspace of $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$.

In Chapter 4 we shall derive a precise characterisation of $\mathbf{C}$, but for now the topological structures of $\mathbf{P}$ and $\mathbf{C}$ so far described will allow us to prove that ESCPs are topological circle planes.

In particular, since $\mathbb{P}_3\mathbb{R}$ is Hausdorff ([8], Proposition 14.5), we have that $\mathbf{C}$ is Hausdorff as well.

3.3 Continuity of the Joining Map

The joining map associated with $(\mathbf{P}, \mathbf{C})$ becomes

$$\alpha : (\mathbf{P}^3)_* \to \mathbb{P}_3\mathbb{R} : (\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto [(a,b,c,d)],$$

where $E(a,b,c,d)$ is the unique plane in $\mathbb{R}^3$ through $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

Proposition 3.3.1. *The joining map associated with an embeddable spherical circle plane* $(\mathbf{P}, \mathbf{C})$ *is continuous.*

Proof. Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be mutually distinct points in $\mathbf{P}$. Then $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are not collinear (by the strict convexity of $\mathbf{P}$), so $E(a, b, c, d)$ is given by

$$(a, b, c) = (\mathbf{x} - \mathbf{y}) \boxtimes (\mathbf{x} - \mathbf{z}) \quad \text{and}$$
$$d = ax_1 + bx_2 + cx_3,$$

where $\boxtimes$ denotes the cross-product in $\mathbb{R}^3$. Hence (a, b, c, d) depends continuously on $(\mathbf{x}, \mathbf{y}, \mathbf{z})$, so the map

$$\widetilde{j} : (\mathbf{P}^3)_* \to \mathbb{R}^4 \backslash \mathbf{0} : (\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto (a, b, c, d)$$

is continuous. Whence the joining map $\alpha = \pi_3 \circ \widetilde{j}$ is continuous. □

3.4 Stability of the Intersection Map

The set

$$\Delta = \left\{ \left([(a, b, c, d)], [(a, b, c, d')] \right) : (a, b, c) \in \mathbb{R}^3 \backslash \mathbf{0};\ d, d' \in \mathbb{R} \right\}$$

corresponds to the pairs of planes in $\mathbb{R}^3$ that do not intersect in a line, *i.e.*, they are either the same plane or are parallel. Hence $(\mathbb{P}_3\mathbb{R} \backslash \mathbf{p_o})^2 \backslash \Delta$ corresponds to the pairs of distinct planes that intersect (necessarily in a line).

Under our identification of circles with planes that non-trivially intersect $\mathbf{P}$, the space $(\mathbf{C}^2)_*$ of pairs of circles that intersect at precisely two points corresponds to pairs of planes whose line of intersection intersects $\mathbf{P}$ at precisely two points (equivalently, intersects $\mathbf{B}$). That is,

$$(\mathbf{C}^2)_* = \{(E, F) \in (\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})^2\backslash\Delta : |E \cap F \cap \mathbf{P}| = 2\}.$$

We subsequently endow $(\mathbf{C}^2)_*$ with the subspace topology of $(\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})^2\backslash\Delta$.

3.4.1 The Set of Intersecting Circles is Open

We proceed to show that $(\mathbf{C}^2)_*$ is open in $\mathbf{C} \times \mathbf{C}$. Let

$$\mathbb{A} := (\mathbb{R}^3\backslash\mathbf{0}) \times \mathbb{R}.$$

Proposition 3.4.1. *The restriction-corestriction*

$$\begin{aligned} \widetilde{\pi}_3 &: \mathbb{A} \to \mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o} \\ &: \mathbf{x} \times d \mapsto [\mathbf{x} \times d] \end{aligned}$$

of the canonical quotient map $\pi_3 : \mathbb{R}^4\backslash\mathbf{0} \to \mathbb{P}_3\mathbb{R}$ is a quotient map.

Proof. The subset

$$\mathbb{A} = \mathbb{R}^4\backslash\{\mathbf{0} \times k : k \in \mathbb{R}\}$$

is open in $\mathbb{R}^4\backslash\mathbf{0}$. Furthermore, $\mathbb{A} = \pi_3^{-1}(\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})$, so $\pi_3^{-1}(\pi_3(\mathbb{A})) = \mathbb{A}$ and $\mathbb{A}$ is saturated[1]. It follows from Theorem A.1.3 that the restriction-corestriction $\widetilde{\pi}_3$ of π_3 is a quotient map. $\square$

Lemma 3.4.2. *The subset*

$$(\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})^2\backslash\Delta$$

[1] A subset $A \subseteq X$ is *saturated* with respect to the quotient map $p : X \to Y$ if $A = p^{-1}\big(p(A)\big)$.

is open in the product space $(\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})^2$*.*

Proof. It suffices to show that Δ is closed in $(\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})^2$. Let

$$\begin{aligned} p : \mathbb{A} &\to \mathbb{R}^3\backslash\mathbf{0} \\ : \mathbf{x} \times d &\mapsto \mathbf{x} \end{aligned}$$

be the continuous projection map. Let $\widetilde{p} : \mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o} \to \mathbb{P}_2\mathbb{R}$ be the map $[\mathbf{x}\times d] \mapsto [\mathbf{x}]$; note that $\widetilde{p}$ is well-defined since

$$[k\mathbf{x} \times kd] = [m\mathbf{x} \times md] \Longleftrightarrow [k\mathbf{x}] = [m\mathbf{x}],$$

for $d, k, m \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^3\backslash\mathbf{0}$. From the commutative diagram

$$\begin{tikzcd} \mathbf{x} \times d \arrow[r, mapsto, "p"] \arrow[d, mapsto, "\widetilde{\pi}_3"'] & \mathbf{x} \arrow[d, mapsto, "\pi_2"] \\ {[\mathbf{x} \times d]} \arrow[r, mapsto, "\widetilde{p}"'] & {[\mathbf{x}]} \end{tikzcd}$$

and Theorem A.1.2, we see that $\widetilde{p}$ is continuous and hence

$$\widetilde{p} \times \widetilde{p} : (\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})^2 \to (\mathbb{P}_2\mathbb{R})^2$$

is continuous. Now, since $\mathbb{P}_2\mathbb{R}$ is Hausdorff, the set

$$\widetilde{\Delta} = \{([\mathbf{x}], [\mathbf{x}]) : [\mathbf{x}] \in \mathbb{P}_2\mathbb{R}\}$$

is a closed subset of $(\mathbb{P}_2\mathbb{R})^2$ (cf. Theorem A.1.11). Thus

$$\Delta = (\widetilde{p} \times \widetilde{p})^{-1}(\widetilde{\Delta})$$

is closed in $(\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})^2$. □

We now give a topological formalism for the intersection of lines in $\mathbb{R}^3$.

Since a line in $\mathbb{R}^3$ is uniquely determined by a point on the line and a direction, it is natural to topologise the set of lines $\mathbf{L}$ as the quotient space of the product space $\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}$ as follows. Recall that we denote a line in $\mathbb{R}^3$ through a point $\mathbf{x}$ and with direction $\mathbf{u} \in \mathbb{S}^2_1$ by $L(\mathbf{x}, [\mathbf{u}])$. Formally, we define the quotient map

$$L : \mathbb{R}^3 \times \mathbb{P}_2\mathbb{R} \to \mathbf{L},$$

which performs the identification

$$L(\mathbf{p}, [\mathbf{x}]) = L(\mathbf{q}, [\mathbf{y}]) \Leftrightarrow \left\{[\mathbf{x}] = [\mathbf{y}] \text{ and } \mathbf{q} = \mathbf{p} + t\mathbf{x} \text{ for some } t \in \mathbb{R}.\right\}$$

Let $E = [\mathbf{x} \times d]$, $F = [\mathbf{y} \times e]$ be two distinct, intersecting planes[2] in $\mathbb{R}^3$. On their line of intersection, the closest point $\mathbf{b}$ to the origin is equal to the intersection point of the planes E, F and the plane $[\mathbf{x} \boxtimes \mathbf{y} \times 0]$ through the origin with normal $\mathbf{x} \boxtimes \mathbf{y}$. The point $\mathbf{b}$ is thus given by[3]

$$\mathbf{b} = \frac{(e\mathbf{x} - d\mathbf{y}) \boxtimes (\mathbf{x} \boxtimes \mathbf{y})}{||\mathbf{x} \boxtimes \mathbf{y}||^2}.$$

Since a line $L(\mathbf{p}, [\mathbf{x}])$ intersects $\mathbf{B}$ if and only if there is a $t \in \mathbb{R}$ such that $\mathbf{p} + t\mathbf{x} \in \mathbf{B}$, the subspace $\mathbf{L_B}$ of $\mathbf{L}$, consisting of all the lines in $\mathbb{R}^3$ that intersect $\mathbf{B}$, is given by

$$\mathbf{L_B} = L(\mathbf{B} \times \mathbb{P}_2\mathbb{R}).$$

Define the *plane intersection* map

$$\chi : (\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})^2\backslash\Delta \to \mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}$$

[2]Notice that we speak of planes in $\mathbb{R}^3$ and points of $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$ interchangeably.
[3]See Goldman [1].

by

$$([\mathbf{x} \times d], [\mathbf{y} \times e]) \mapsto \left(\frac{(e\mathbf{x} - d\mathbf{y}) \boxtimes (\mathbf{x} \boxtimes \mathbf{y})}{||\mathbf{x} \boxtimes \mathbf{y}||^2}, [\mathbf{x} \boxtimes \mathbf{y}] \right).$$

Then

$$\psi := L \circ \chi : (\mathbb{P}_3\mathbb{R} \backslash \mathbf{p_o})^2 \backslash \Delta \to \mathbf{L}$$

maps a pair of distinct intersecting planes in $\mathbb{R}^3$ to their line of intersection. We shall establish the continuity of ψ in order to deduce that $(\mathbf{C}^2)_*$ is open in $\mathbf{C}^2$; to show that ψ is continuous it suffices to show that χ is continuous.

Lemma 3.4.3. *The plane intersection map, χ, is continuous.*

Proof. Denote the first and second coordinate functions of χ by

$$\chi_1 : (\mathbb{P}_3\mathbb{R} \backslash \mathbf{p_o})^2 \backslash \Delta \to \mathbb{R}^3 \quad \text{and}$$
$$\chi_2 : (\mathbb{P}_3\mathbb{R} \backslash \mathbf{p_o})^2 \backslash \Delta \to \mathbb{P}_2\mathbb{R},$$

respectively.

Since $\mathbb{A}$ and $\widetilde{\pi}_3(\mathbb{A}) = \mathbb{P}_3\mathbb{R} \backslash \mathbf{p_o}$ are locally compact and Hausdorff (being open subsets of locally compact, Hausdorff spaces; cf. Theorem A.1.7), by Theorem A.1.5 we have that

$$\widetilde{\pi}_3 \times \widetilde{\pi}_3 : \mathbb{A}^2 \to (\mathbb{P}_3\mathbb{R} \backslash \mathbf{p_o})^2,$$

where $\widetilde{\pi}_3$ is as defined in Proposition 3.4.1, is a quotient map.

Let $\lambda_1 : \mathbb{A}^2 \to \mathbb{R}^3$ be the map given by

$$\lambda_1 : (\mathbf{x} \times d, \mathbf{y} \times e) \mapsto \frac{(e\mathbf{x} - d\mathbf{y}) \boxtimes (\mathbf{x} \boxtimes \mathbf{y})}{||\mathbf{x} \boxtimes \mathbf{y}||^2}.$$

Then λ_1 is well-defined, and is continuous by the continuity of the cross-product. Then, since

$$\chi_1 \circ (\widetilde{\pi}_3 \times \widetilde{\pi}_3) = \lambda_1,$$

as in the commutative diagram:

$$\begin{tikzcd}
(\mathbf{x} \times d, \mathbf{y} \times e) \arrow[r, mapsto, "\lambda_1"] \arrow[d, mapsto, "\widetilde{\pi}_3 \times \widetilde{\pi}_3"'] & \frac{(e\mathbf{x}-d\mathbf{y}) \boxtimes (\mathbf{x} \boxtimes \mathbf{y})}{||\mathbf{x} \boxtimes \mathbf{y}||^2} \\
([\mathbf{x} \times d], [\mathbf{y} \times e]) \arrow[ur, mapsto, "\chi_1"'] &
\end{tikzcd}$$

we obtain from Theorem A.1.1 that the coordinate function χ_1 is continuous.

Let $\lambda_2 : \mathbb{A}^2 \to \mathbb{P}_2\mathbb{R}$ be the map given by the composition of the projection, cross product and π_2 as follows:

$$\lambda_2 : (\mathbf{x} \times d, \mathbf{y} \times e) \mapsto (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \boxtimes \mathbf{y} \mapsto [\mathbf{x} \boxtimes \mathbf{y}].$$

As each map in this composition is continuous, so is λ_2. Hence, since

$$\chi_2 \circ (\widetilde{\pi}_3 \times \widetilde{\pi}_3) = \lambda_2,$$

as in the commutative diagram:

$$\begin{tikzcd}
(\mathbf{x} \times d, \mathbf{y} \times e) \arrow[r, mapsto, "\lambda_2"] \arrow[d, mapsto, "\widetilde{\pi}_3 \times \widetilde{\pi}_3"'] & {[\mathbf{x} \boxtimes \mathbf{y}]} \\
([\mathbf{x} \times d], [\mathbf{y} \times e]) \arrow[ur, mapsto, "\chi_2"'] &
\end{tikzcd}$$

we obtain from Theorem A.1.1 that the second coordinate function χ_2 is continuous.

This shows that χ is continuous. □

Thus we have:

Proposition 3.4.4. *The map* $\psi : (\mathbb{P}_3\mathbb{R} \backslash \mathbf{p_o})^2 \backslash \Delta \to \mathbf{L}$ *defined by*

$$\psi = L \circ \chi : ([\mathbf{x} \times d], [\mathbf{y} \times e]) \mapsto L\left(\frac{(e\mathbf{x}-d\mathbf{y}) \boxtimes (\mathbf{x} \boxtimes \mathbf{y})}{||\mathbf{x} \boxtimes \mathbf{y}||^2}, [\mathbf{x} \boxtimes \mathbf{y}]\right)$$

is continuous.

Now, from the definitions of $(\mathbf{C}^2)_*$, $\mathbf{L_B}$ and the construction of ψ, we have that $(E, F) \in (\mathbf{C}^2)_*$ if and only if $\psi(E, F) \in \mathbf{L_B}$, so

$$(\mathbf{C}^2)_* = \psi^{-1}(\mathbf{L_B}).$$

The next step, then, is to show that $\mathbf{L_B}$ is open in $\mathbf{L}$.

Proposition 3.4.5. *The set $\mathbf{L_B}$ of lines intersecting $\mathbf{B}$ is an open subset of the space $\mathbf{L}$ of all lines in $\mathbb{R}^3$.*

Proof. By the definition of $\mathbf{L}$, it suffices to show that $L^{-1}(\mathbf{L_B})$ is open in $\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}$. Let $(\mathbf{p}, [\mathbf{x}]) \in L^{-1}(\mathbf{L_B})$. If $\mathbf{p} \in \mathbf{B}$ then there is an $r > 0$ such that $\mathcal{B}_{\rho_3}(\mathbf{p}, r) \subseteq \mathbf{B}$. Hence

$$U := \mathcal{B}_{\rho_3}(\mathbf{p}, r) \times \mathbb{P}_2\mathbb{R}$$

is an open subset in $\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}$ containing $(\mathbf{p}, [\mathbf{x}])$. Furthermore, $L(\mathbf{q}, [\mathbf{y}]) \in \mathbf{L_B}$ for each $\mathbf{q} \in \mathcal{B}_{\rho_3}(\mathbf{p}, r)$ and any $[\mathbf{y}] \in \mathbb{P}_2\mathbb{R}$, since

$$\mathbf{q} + 0\mathbf{y} = \mathbf{q} \in \mathcal{B}_{\rho_3}(\mathbf{p}, r) \subseteq \mathbf{B}.$$

Whence $L(U) \subseteq \mathbf{L_B}$, so $U \subseteq L^{-1}(\mathbf{L_B})$.

Now suppose that $\mathbf{p} \notin \mathbf{B}$. Then $\mathbf{b} = \mathbf{p} + t\mathbf{x} \in \mathbf{B}$ for some $t \neq 0$; choose $r > 0$ so that $\mathcal{B}_{\rho_3}(\mathbf{b}, r) \subseteq \mathbf{B}$. Let

$$U := \mathcal{B}_{\rho_3}(\mathbf{p}, R) \times \mathcal{B}_{\varrho_2}([\mathbf{x}], \sqrt{2(1 - \cos\theta_0)}\,),$$

where

$$R := \tfrac{1}{2}||\mathbf{p} - \mathbf{b}|| \sin\theta_0, \quad \text{and} \quad \theta_0 := \arctan \tfrac{r}{2||\mathbf{p}-\mathbf{b}||}.$$

Then U is an open subset of $\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}$ containing $(\mathbf{p}, [\mathbf{x}])$; we show that $U \subseteq$

$L^{-1}(\mathbf{L_B})$. Let us first explain the choice of the radius $\sqrt{2(1-\cos\theta_0)}$ with the aid of Figure 3.4.1.1, which illustrates the various objects projected onto the plane $[\mathbf{p} \boxtimes \mathbf{b}]^\perp$ in the generic case where $\mathbf{p}$ and $\mathbf{b}$ are linearly independent (one can achieve this by changing the origin).[4]

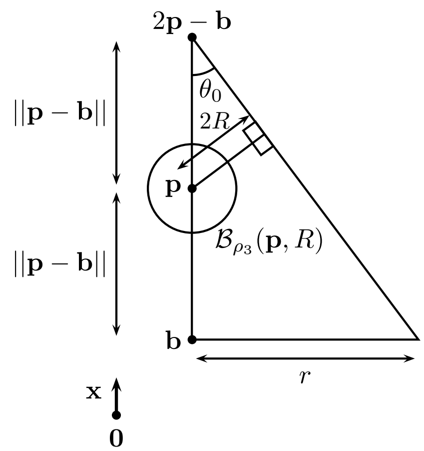


Figure 3.4.1.1

Let $[\mathbf{y}] \in \mathcal{B}_{\varrho_2}([\mathbf{x}], \sqrt{2(1-\cos\theta_0)}\,)$; so

$$\varrho_2([\mathbf{x}],[\mathbf{y}]) = \min\left\{\left\|\tfrac{\mathbf{x}}{||\mathbf{x}||}+\tfrac{\mathbf{y}}{||\mathbf{y}||}\right\|, \left\|\tfrac{\mathbf{x}}{||\mathbf{x}||}-\tfrac{\mathbf{y}}{||\mathbf{y}||}\right\|\right\} < \sqrt{2(1-\cos\theta_0)}.$$

Let $\theta = \arccos\frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||||\mathbf{y}||}$; that is, the angle in $[0, \frac{\pi}{2}]$ between the subspaces $[\mathbf{x}]$ and $[\mathbf{y}]$. Referring to Figure 3.4.1.2 and applying the cosine law, we have

$$\begin{aligned}\cos\theta = \frac{1^2+1^2-(\varrho_2([\mathbf{x}],[\mathbf{y}]))^2}{2(1)(1)} &= \frac{2-(\varrho_2([\mathbf{x}],[\mathbf{y}]))^2}{2}\\ &> \frac{2-\sqrt{2(1-\cos\theta_0)}^2}{2}\\ &= \cos\theta_0,\end{aligned}$$

so $\theta < \theta_0$.

[4]If W is a linear subspace of $\mathbb{R}^d$ then $W^\perp$ denotes its orthogonal subspace.

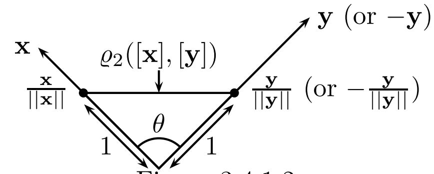


Figure 3.4.1.2

Thus the direction $[\mathbf{y}]$ of a line $L(\mathbf{q},[\mathbf{y}])$ with

$$[\mathbf{y}] \in \mathcal{B}_{\varrho_2}([\mathbf{x}], \sqrt{2(1-\cos\theta_0)}\,) := V$$

makes an angle smaller than θ_0 with a line with direction $[\mathbf{x}] = [\mathbf{p}-\mathbf{b}]$.

With the aid of Figure 3.4.1.3, let us explain the choice of R and θ_0: a line with basepoint $\mathbf{q}$ in $\mathcal{B}_{\rho_3}(\mathbf{p},R)$ and direction $[\mathbf{y}] \in V$ makes an angle of $\theta < \theta_0$ with $[\mathbf{x}]$ and consequently intersects $\mathcal{B}_{\rho_3}(\mathbf{b},r) \subseteq \mathbf{B}$.

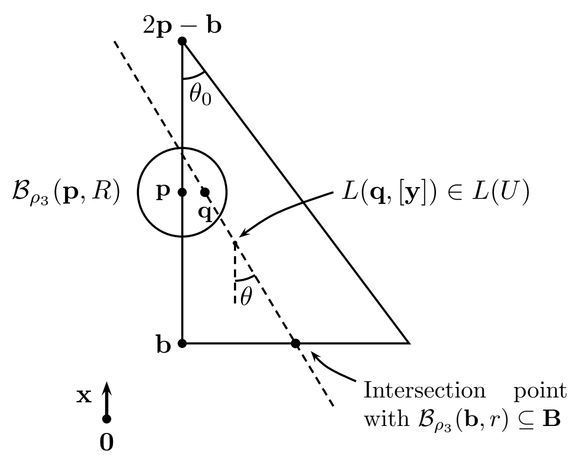


Figure 3.4.1.3

This implies that $U \subseteq L^{-1}(\mathbf{L_B})$. We deduce that $L^{-1}(\mathbf{L_B})$ is open in $\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}$ and hence that $\mathbf{L_B}$ is open in $L(\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}) = \mathbf{L}$. $\square$

Theorem 3.4.6. *The subset of pairs of intersecting circles* $(\mathbf{C}^2)_*$ *is open in* $\mathbf{C}^2$.

Proof. Combining Propositions 3.4.4 and 3.4.5, we have that $(\mathbf{C}^2)_* = \psi^{-1}(\mathbf{L_B})$

is open in $(\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})^2\backslash\Delta$. Thus, as $(\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})^2\backslash\Delta$ is open in $(\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})^2$ (Lemma 3.4.2), we further see that $(\mathbf{C}^2)_*$ is open in $(\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})^2$. As $(\mathbf{C}^2)_*$ is contained in $\mathbf{C}^2$, this means that $(\mathbf{C}^2)_*$ is open in the subspace $\mathbf{C}^2$ of $(\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o})^2$. $\square$

3.4.2 Continuity of the Intersection Map

For a line $L(\mathbf{p},[\mathbf{x}]) \in \mathbf{L}$, if $\mathbf{q} = \mathbf{p} + t\mathbf{x}$ for some $t \in \mathbb{R}$ we write $\mathbf{q} \in L(\mathbf{p},[\mathbf{x}])$. We extend this notation so that for a subset $A \subseteq \mathbb{R}^3$, we write

$$L(\mathbf{p},[\mathbf{x}]) \cap A := \{\mathbf{p} \in \mathbb{R}^3 : \mathbf{p} \in L(\mathbf{p},[\mathbf{x}]) \text{ and } \mathbf{p} \in A\}.$$

Let

$$(\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R})_* := \{(\mathbf{p},[\mathbf{x}]) \in \mathbb{R}^3 \times \mathbb{P}_2\mathbb{R} : |L(\mathbf{p},[\mathbf{x}]) \cap \mathbf{P}| = 2\}$$

denote the set of lines of $\mathbb{R}^3$ that intersect $\mathbf{P}$ at precisely two points. We may then decompose the intersection map γ as

$$\gamma : (\mathbf{C}^2)_* \xrightarrow{\chi|(\mathbf{C}^2)_*} (\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R})_* \xrightarrow{\mu} \widetilde{\mathbf{P}^2},$$

where μ maps each point $(\mathbf{p},[\mathbf{x}])$ of $(\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R})_*$ to the (unordered) pair of points of intersection $L(\mathbf{p},[\mathbf{x}]) \cap \mathbf{P}$. Hence to complete the proof that γ is continuous we need to show that μ is continuous.

We proceed to determine the metric structures of the domain and codomain of μ.

The product topology of $\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}$ is precisely the topology induced by the product metric (cf. [4], pp. 67)

$$\sigma : (\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R})^2 \to \mathbb{R}_{\geq 0}$$

given by

$$\sigma\big((\mathbf{p},[\mathbf{x}]),(\mathbf{q},[\mathbf{y}])\big) = ||\mathbf{p}-\mathbf{q}|| + \varrho_2([\mathbf{x}],[\mathbf{y}]).$$

It follows that a sequence of points $\big((\mathbf{p}_n,[\mathbf{x}_n])\big)$ in $\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}$ converges to the point $(\mathbf{p},[\mathbf{x}])$ if and only if $\mathbf{p}_n \to \mathbf{p}$ and $[\mathbf{x}_n] \to [\mathbf{x}]$. Since a line $L(\mathbf{p},[\mathbf{x}]) \in \mathbf{L}$ is uniquely determined by any basepoint $\mathbf{q} \in \{\mathbf{p}+t\mathbf{x} : t \in \mathbb{R}\}$ and a direction $[\mathbf{x}]$, if $\big((\mathbf{p}_n,[\mathbf{x}_n])\big) \to (\mathbf{p},[\mathbf{x}])$ we may therefore say that the sequence of lines $K_n := L(\mathbf{p}_n,[\mathbf{x}_n])$ *converges to* $K := L(\mathbf{p},[\mathbf{x}])$, and write $K_n \to K$ (see Figure 3.4.2.1).

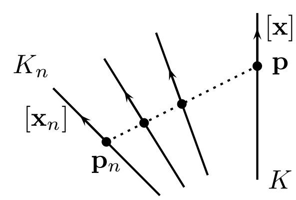


Figure 3.4.2.1: Lines converging.

If $(\mathbf{a}_n)$ is a sequence in $\mathbb{R}^3$ such that $\mathbf{a}_n \in K_n := L(\mathbf{p}_n,[\mathbf{x}_n])$ for each n, we write $(\mathbf{a}_n) \in (K_n)$.

Let the topology on $\widetilde{\mathbf{P}^2} = \big\{\{\mathbf{x},\mathbf{y}\} : \mathbf{x},\mathbf{y} \in \mathbf{P}\big\}$ be the quotient topology on $\mathbf{P}^2$ defined by the quotient map Ω, which performs the identification

$$\Omega(\mathbf{x},\mathbf{y}) = \Omega(\mathbf{u},\mathbf{v}) \Leftrightarrow \big\{(\mathbf{x},\mathbf{y}) = (\mathbf{u},\mathbf{v}) \text{ or } (\mathbf{x},\mathbf{y}) = (\mathbf{v},\mathbf{u})\big\}.$$

We show that this is precisely the topology induced by the metric

$$\beta : \widetilde{\mathbf{P}^2} \times \widetilde{\mathbf{P}^2} \to \mathbb{R}_{\geq 0}$$

defined by

$$\beta(\{\mathbf{x},\mathbf{y}\},\{\mathbf{u},\mathbf{v}\}) = \min\big\{||\mathbf{x}-\mathbf{u}|| + ||\mathbf{y}-\mathbf{v}||, ||\mathbf{x}-\mathbf{v}|| + ||\mathbf{y}-\mathbf{u}||\big\}.$$

We first show that, for each $(\mathbf{x}, \mathbf{y}) \in \mathbf{P}^2$,

$$\Omega\big(\mathcal{B}_\rho\big((\mathbf{x}, \mathbf{y}), r\big)\big) = \mathcal{B}_\beta\big(\Omega(\mathbf{x}, \mathbf{y}), r\big),$$

where $\rho := \rho_6$ is the Euclidean metric on $(\mathbb{R}^3)^2$. Let $(\mathbf{u}, \mathbf{v}) \in \mathcal{B}_\rho\big((\mathbf{x}, \mathbf{y}), r\big)$. Then

$$\begin{aligned}\beta\big(\Omega(\mathbf{x}, \mathbf{y}), \Omega(\mathbf{u}, \mathbf{v})\big) &= \min\big\{||(\mathbf{x}, \mathbf{y}) - (\mathbf{u}, \mathbf{v})||, ||(\mathbf{x}, \mathbf{y}) - (\mathbf{v}, \mathbf{u})||\big\} \\ &\leq ||(\mathbf{x}, \mathbf{y}) - (\mathbf{u}, \mathbf{v})|| < r,\end{aligned}$$

so $\Omega(\mathbf{u}, \mathbf{v}) \in \mathcal{B}_\beta\big(\Omega(\mathbf{x}, \mathbf{y}), r\big)$. Hence $\Omega\big(\mathcal{B}_\rho\big((\mathbf{x}, \mathbf{y}), r\big)\big) \subseteq \mathcal{B}_\beta\big(\Omega(\mathbf{x}, \mathbf{y}), r\big)$.

Conversely, let $\{\mathbf{u}, \mathbf{v}\} \in \mathcal{B}_\beta\big(\Omega(\mathbf{x}, \mathbf{y}), r\big)$; then

$$\min\big\{||(\mathbf{x}, \mathbf{y}) - (\mathbf{u}, \mathbf{v})||, ||(\mathbf{x}, \mathbf{y}) - (\mathbf{v}, \mathbf{u})||\big\} < r$$

and so $||(\mathbf{x}, \mathbf{y}) - (\mathbf{u}, \mathbf{v})|| < r$ or $||(\mathbf{x}, \mathbf{y}) - (\mathbf{v}, \mathbf{u})||) < r$. That is, either $(\mathbf{u}, \mathbf{v})$ or $(\mathbf{v}, \mathbf{u})$ lies in $\mathcal{B}_\rho\big((\mathbf{x}, \mathbf{y})\big)$, so

$$\{\mathbf{u}, \mathbf{v}\} = \Omega(\mathbf{u}, \mathbf{v}) = \Omega(\mathbf{v}, \mathbf{u}) \in \Omega\big(\mathcal{B}_\rho\big((\mathbf{x}, \mathbf{y}), r\big)\big).$$

Hence

$$\mathcal{B}_\beta\big(\Omega(\mathbf{x}, \mathbf{y}), r\big) \subseteq \Omega\big(\mathcal{B}_\rho\big((\mathbf{x}, \mathbf{y}), r\big)\big)$$

and we obtain that $\mathcal{B}_\beta\big(\Omega(\mathbf{x}, \mathbf{y}), r\big) = \Omega\big(\mathcal{B}_\rho\big((\mathbf{x}, \mathbf{y}), r\big)\big)$.

Now, a subset U is open in $\widetilde{\mathbf{P}^2}$ with respect to the quotient topology if and only if $\Omega^{-1}(U)$ is open in $\mathbf{P}^2$ with respect to the topology induced by ρ; that is, if and only if for each $(\mathbf{x}, \mathbf{y}) \in \Omega^{-1}(U)$, there is an $r > 0$ such that $\mathcal{B}_\rho\big((\mathbf{x}, \mathbf{y}), r\big) \subseteq \Omega^{-1}(U)$. This holds if and only if for each $\Omega(\mathbf{x}, \mathbf{y}) \in U$, there is an $r > 0$ such that

$$\Omega\big(\mathcal{B}_\rho\big((\mathbf{x},\mathbf{y}),r\big)\big) = \mathcal{B}_\beta\big((\mathbf{x},\mathbf{y}),r\big) \subseteq U;$$

that is, if and only if U is open with respect to the topology induced by β.

Thus the quotient topology on $\widetilde{\mathbf{P}^2}$ is precisely the topology induced by the metric β. Therefore, for a sequence $(\{\mathbf{x}_n,\mathbf{y}_n\})$ in $\widetilde{\mathbf{P}^2}$, we have

$$(\{\mathbf{x}_n,\mathbf{y}_n\}) \to \{\mathbf{x},\mathbf{y}\} \text{ if and only if } \beta\big((\mathbf{x}_n,\mathbf{y}_n),(\mathbf{x},\mathbf{y})\big)\to 0.$$

We now gather some information about the relationships between lines and points on them, and between lines and points on their intersections with planes.

Lemma 3.4.7. *Let* $(\mathbf{p}_n,[\mathbf{x}_n])$ *be a sequence in* $\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}$ *converging to the point* $(\mathbf{p},[\mathbf{x}]) \in \mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}$. *For each* n, *let* $K_n = L(\mathbf{p}_n,[\mathbf{x}_n])$; *put* $K = L(\mathbf{p},[\mathbf{x}])$, *so* $K_n \to K$. *If* $\mathbf{a} \in K$, *then there is a sequence* $(\mathbf{a}_n) \in (K_n)$ *such that* $\mathbf{a}_n \to \mathbf{a}$ *in* $\mathbb{R}^3$.

Proof. Let $\mathbf{a} = \mathbf{p} + t\mathbf{x}$ for some $t \in \mathbb{R}$. Let $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$; for each n, let $\mathbf{u}_n = \frac{\mathbf{x}_n}{||\mathbf{x}_n||}$ and choose $\widetilde{\mathbf{u}}_n \in \{\mathbf{u}_n, -\mathbf{u}_n\}$ so that

$$\|\widetilde{\mathbf{u}}_n - \mathbf{u}\| = \min\left\{\|\mathbf{u}_n + \mathbf{u}\|, \|\mathbf{u}_n - \mathbf{u}\|\right\};$$

thence $\widetilde{\mathbf{u}}_n \to \mathbf{u}$. Define the sequence $(\mathbf{a}_n)$ in $\mathbb{R}^3$ by

$$\mathbf{a}_n = \mathbf{p}_n + t||\mathbf{x}||\widetilde{\mathbf{u}}_n.$$

Then $(\mathbf{a}_n) \in (\mathbf{p}_n,[\mathbf{x}_n])$ and

$$\mathbf{a}_n \to \mathbf{p} + t||\mathbf{x}||\mathbf{u} = \mathbf{p} + t\mathbf{x} = \mathbf{a}. \qquad \square$$

Lemma 3.4.8. *Let* $(\mathbf{p}_n, [\mathbf{x}_n])$ *be a sequence in* $\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}$ *converging to the point* $(\mathbf{p}, [\mathbf{x}])$. *Let* E *be a plane in* $\mathbb{R}^3$ *that is orthogonal to the line* $K := L(\mathbf{p}, [\mathbf{x}])$. *Then there is an* $N > 0$ *such that* $K_n := L(\mathbf{p}_n, [\mathbf{x}_n])$ *intersects* E *at a unique point for all* $n > N$.

Proof. Choose $\mathbf{p}$, $\mathbf{q} \in K$ so that they lie in distinct open half-spaces determined by E. Since $K_n \to K$, by Lemma 3.4.7 there are sequences $(\mathbf{p}_n), (\mathbf{q}_n) \in (K_n)$ such that $\mathbf{p}_n \to \mathbf{p}$ and $\mathbf{q}_n \to \mathbf{q}$.

Firstly, suppose that $K_n \cap E = \varnothing$ for infinitely many n. Then, since E separates $\mathbb{R}^3$, there is a closed half-space $\overline{H}$ determined by E and a subsequence (K_{n_k}) of (K_n) such that $K_{n_k} \subseteq \overline{H}$ for all k. But then $\mathbf{p}_{n_k}$ and $\mathbf{q}_{n_k}$ lie in $K_{n_k} \subseteq \overline{H}$ for all k and so, since $\mathbf{p}_{n_k} \to \mathbf{p}$ and $\mathbf{q}_{n_k} \to \mathbf{q}$, we have that $\mathbf{p}$, $\mathbf{q} \in \overline{H}$ — contradicting that $\mathbf{p}$ and $\mathbf{q}$ lie on opposite sides of E.

Similarly, if $K_n \subseteq E$ for infinitely many n, then there is a subsequence (K_{n_k}) of (K_n) such that $K_{n_k} \subseteq E$ for all k. So, as $\mathbf{p}_{n_k} \to \mathbf{p}$, we have that $\mathbf{p} \in E$ (as E is closed in $\mathbb{R}^3$) — contradicting that $\mathbf{p}$ does not lie on E.

Thus there is an $N > 0$ such that, for all $n > N$: K_n does not lie on E and is not parallel to E, so intersects E at a unique point. □

From Lemma 3.4.8 we may formulate the following.

Lemma 3.4.9. *Let* $(\mathbf{p}, [\mathbf{x}]) \in \mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}$ *and let* E *be a plane in* $\mathbb{R}^3$ *that is orthogonal to the line* $K := L(\mathbf{p}, [\mathbf{x}])$. *Let* $(\mathbf{p}_n, [\mathbf{x}_n])$ *be a sequence in* $\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R}$ *converging to* $(\mathbf{p}, [\mathbf{x}])$ *such that, for each* n, $K_n := L(\mathbf{p}_n, [\mathbf{x}_n])$ *intersects* E *at a unique point* $\mathbf{c}_n$. *Then* $\mathbf{c}_n \to \mathbf{c}$, *where* $\mathbf{c} = K \cap E$.

Proof. Let $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$. For each n, let $\mathbf{u}_n = \frac{\mathbf{x}_n}{||\mathbf{x}_n||}$ and choose $\widetilde{\mathbf{u}}_n \in \{\mathbf{u}_n, -\mathbf{u}_n\}$ so that

$$\|\widetilde{\mathbf{u}}_n - \mathbf{u}\| = \min\{\|\mathbf{u}_n + \mathbf{u}\|, \|\mathbf{u}_n - \mathbf{u}\|\};$$

thence $\widetilde{\mathbf{u}}_n \to \mathbf{u}$.

Put $E = \{\mathbf{y} \in \mathbb{R}^3 : \mathbf{y} \cdot \mathbf{x} + d = 0\}$ for some $d \in \mathbb{R}$. For a line $L(\mathbf{q}, [\mathbf{y}])$ intersecting E, the point of intersection $\mathbf{b} := L(\mathbf{q}, [\mathbf{y}]) \cap E$ is found by solving the system

$$\begin{cases} \mathbf{b} \cdot \mathbf{x} + d = 0 \\ \mathbf{b} = \mathbf{q} + t\mathbf{y}. \end{cases}$$

Substituting the latter equation into the former, we have $(\mathbf{q} + t\mathbf{y}) \cdot \mathbf{x} + d = 0$, so $t = -\frac{\mathbf{q}\cdot\mathbf{x}+d}{\mathbf{y}\cdot\mathbf{x}}$; hence

$$\mathbf{b} = \mathbf{q} - \frac{\mathbf{q} \cdot \mathbf{x} + d}{\mathbf{y} \cdot \mathbf{x}}\, \mathbf{y}.$$

Furthermore, the mapping

$$(\mathbf{q}, [\mathbf{y}]) \mapsto L(\mathbf{q}, [\mathbf{y}]) \cap E$$

is well-defined since, for $(\mathbf{q} + s\mathbf{y}, [k\mathbf{y}]) \in L(\mathbf{q}, [\mathbf{y}])$, with $s \in \mathbb{R}$ and $k \in \mathbb{R}\backslash 0$, we have

$$\begin{aligned} (\mathbf{q} + s\mathbf{y}) - \frac{(\mathbf{q} + s\mathbf{y}) \cdot \mathbf{x} + d}{(k\mathbf{y}) \cdot \mathbf{x}}\, (k\mathbf{y}) &= \mathbf{q} + s\mathbf{y} - \frac{\mathbf{q} \cdot \mathbf{x} + s\mathbf{y} \cdot \mathbf{x} + d}{\mathbf{y} \cdot \mathbf{x}}\, \mathbf{y} \\ &= \mathbf{q} + s\mathbf{y} - \frac{\mathbf{q} \cdot \mathbf{x} + d}{\mathbf{y} \cdot \mathbf{x}}\, \mathbf{y} - s\frac{\mathbf{y} \cdot \mathbf{x}}{\mathbf{y} \cdot \mathbf{x}}\, \mathbf{y} \\ &= \mathbf{b}. \end{aligned}$$

Thus $\mathbf{c}_n$ is given by

$$\begin{aligned} \mathbf{c}_n &= L(\mathbf{p}_n, [\mathbf{x}_n]) \cap E = L(\mathbf{p}_n, [\widetilde{\mathbf{u}}_n]) \cap E \\ &= \mathbf{p}_n - \frac{\mathbf{p}_n \cdot \mathbf{u} + d}{\widetilde{\mathbf{u}}_n \cdot \mathbf{u}}\, \widetilde{\mathbf{u}}_n, \end{aligned}$$

so

$$\begin{aligned}\mathbf{c}_n &\to \mathbf{p} - \frac{\mathbf{p}\cdot\mathbf{u} + d}{\widetilde{\mathbf{u}}\cdot\mathbf{u}}\,\widetilde{\mathbf{u}} \\ &= L(\mathbf{p},[\mathbf{u}]) \cap E = L(\mathbf{p},[\mathbf{x}]) \cap E = \mathbf{c}.\end{aligned}$$ □

We also find that our definition of line convergence in $\mathbb{R}^3$ induces continuity on the joining of two points by a line.

Lemma 3.4.10. *[Continuity of joining two points] Let* $(\mathbf{a}_n)$, $(\mathbf{b}_n)$ *be sequences in* $\mathbb{R}^3$ *converging to distinct points* $\mathbf{a}$, $\mathbf{b}$, *respectively. Then* $\mathbf{a}_n \neq \mathbf{b}_n$ *for* n *sufficiently large and the sequence of lines* (K_n), *given by* $K_n = L(\mathbf{a}_n, [\mathbf{a}_n - \mathbf{b}_n])$ *for each* n, *converges to* $K := L(\mathbf{a}, [\mathbf{a} - \mathbf{b}])$.

Proof. It suffices to show that $(\mathbf{a}_n, [\mathbf{a}_n - \mathbf{b}_n]) \to (\mathbf{a}, [\mathbf{a} - \mathbf{b}])$. Since $\mathbf{a}_n \to \mathbf{a}$, we are left to show that $[\mathbf{a}_n - \mathbf{b}_n] \to [\mathbf{a} - \mathbf{b}]$. We have $\mathbf{a}_n - \mathbf{b}_n \to \mathbf{a} - \mathbf{b}$. If $\mathbf{a}_n = \mathbf{b}_n$ for infinitely many n, denote the corresponding subsequences of $(\mathbf{a}_n)$ and $(\mathbf{b}_n)$ by $(\mathbf{a}_{n_k})$ and $(\mathbf{b}_{n_k})$, respectively. Then

$$\mathbf{0} = \mathbf{a}_{n_k} - \mathbf{b}_{n_k} \to \mathbf{a} - \mathbf{b} \neq \mathbf{0}$$

— a contradiction. Hence there is an $N > 0$ such that $\mathbf{a}_n - \mathbf{b}_n \neq \mathbf{0}$ for all $n > N$, so $\frac{\mathbf{a}_n - \mathbf{b}_n}{||\mathbf{a}_n - \mathbf{b}_n||} \to \frac{\mathbf{a} - \mathbf{b}}{||\mathbf{a} - \mathbf{b}||}$ and

$$\begin{aligned}\varrho_2([\mathbf{a}_n - \mathbf{b}_n], [\mathbf{a} - \mathbf{b}]) &= \min\left\{\left\|\tfrac{\mathbf{a}_n - \mathbf{b}_n}{||\mathbf{a}_n - \mathbf{b}_n||} + \tfrac{\mathbf{a} - \mathbf{b}}{||\mathbf{a} - \mathbf{b}||}\right\|, \left\|\tfrac{\mathbf{a}_n - \mathbf{b}_n}{||\mathbf{a}_n - \mathbf{b}_n||} - \tfrac{\mathbf{a} - \mathbf{b}}{||\mathbf{a} - \mathbf{b}||}\right\|\right\} \\ &\leq \left\|\tfrac{\mathbf{a}_n - \mathbf{b}_n}{||\mathbf{a}_n - \mathbf{b}_n||} - \tfrac{\mathbf{a} - \mathbf{b}}{||\mathbf{a} - \mathbf{b}||}\right\| \to 0.\end{aligned}$$

That is, $[\mathbf{a}_n - \mathbf{b}_n] \to [\mathbf{a} - \mathbf{b}]$. □

We are now well-placed to prove that μ is continuous.

Proposition 3.4.11. *The map* $\mu : (\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R})_* \to \widetilde{\mathbf{P}^2}$ *given by*

$$(\mathbf{p}, [\mathbf{x}]) \mapsto L(\mathbf{p}, [\mathbf{x}]) \cap \mathbf{P}$$

is continuous.

The following construction, which shall form part of the proof of Proposition 3.4.11, will be of use later, so we state it formally.

Proposition 3.4.12. *Let* $\big((\mathbf{p}_n, [\mathbf{x}_n])\big)$ *be a sequence in* $(\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R})_*$ *converging to the point* $(\mathbf{p}, [\mathbf{x}])$*. Let* E *be a plane perpendicular to the line* $L(\mathbf{p}, [\mathbf{x}])$*. Then there are planes* E_1 *and* E_2*, parallel to* E *and intersecting* $L(\mathbf{p}, [\mathbf{x}])$ *at two points in distinct components of* $\mathbb{R}^3 \backslash E$*. Moreover, there are closed half-spaces* $\overline{H_i}$ $(i = 1, 2)$ *determined by* E_i*, not containing* E*, such that for sufficiently large* n*,*

$$\widetilde{\mathbf{a}}_n := L(\mathbf{p}_n, [\mathbf{x}_n]) \cap \mathbf{P} \cap H_1,$$
$$\widetilde{\mathbf{b}}_n := L(\mathbf{p}_n, [\mathbf{x}_n]) \cap \mathbf{P} \cap H_2,$$

define sequences $(\widetilde{\mathbf{a}}_n)$*,* $(\widetilde{\mathbf{b}}_n)$*, which satisfy:*

(i) $L(\mathbf{p}_n, [\mathbf{x}_n]) \cap \mathbf{P} = \{\widetilde{\mathbf{a}}_n, \widetilde{\mathbf{b}}_n\}$*,*

(ii) $\widetilde{\mathbf{a}}_n \to \mathbf{a} := L(\mathbf{p}, [\mathbf{x}]) \cap \mathbf{P} \cap \overline{H_1}$*, and*

(iii) $\widetilde{\mathbf{b}}_n \to \mathbf{b} := L(\mathbf{p}, [\mathbf{x}]) \cap \mathbf{P} \cap \overline{H_2}$*.*

Proof of Propositions 3.4.11 and 3.4.12. Let $\big((\mathbf{p}_n, [\mathbf{x}_n])\big)$ be a sequence in $(\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R})_*$ converging to the point $(\mathbf{p}, [\mathbf{x}])$; put $K_n = L(\mathbf{p}_n, [\mathbf{x}_n])$ for each n, and let $K = L(\mathbf{p}, [\mathbf{x}])$. By the definition of $(\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R})_*$, we may let $\{\mathbf{a}, \mathbf{b}\} := K \cap \mathbf{P}$ and $\{\mathbf{a}_n, \mathbf{b}_n\} := K_n \cap \mathbf{P}$ for each n.

We first show that for sufficiently large n, $\mathbf{a}_n$ and $\mathbf{b}_n$ lie in disjoint closed half-spaces of $\mathbb{R}^3$, each containing precisely one of $\mathbf{a}$ or $\mathbf{b}$.

Let E be a plane perpendicular to K that intersects K at a point $\mathbf{c}$ in $]\mathbf{a}, \mathbf{b}[$. Since $]\mathbf{a}, \mathbf{b}[\subseteq \mathbf{B}$, there is an open ball $U \subseteq \mathbf{B}$ containing $\mathbf{c}$. By removing finitely many elements of (K_n) if necessary, from Lemma 3.4.8 we may assume that E intersects each K_n at a unique point. Let $(\mathbf{c}_n)$ be the corresponding sequence; that is, $\mathbf{c}_n = K_n \cap E$ for each n. By Lemma 3.4.9, $\mathbf{c}_n \to \mathbf{c}$ and so by removing finitely many elements of $(\mathbf{c}_n)$ if necessary we may further assume that $\mathbf{c}_n \in U$ for each n.

Let E_1, E_2 be planes parallel to E chosen so that

$$E_1 \cap K \subseteq]\mathbf{a}, \mathbf{c}[\cap U \text{ and}$$
$$E_2 \cap K \subseteq]\mathbf{b}, \mathbf{c}[\cap U,$$

as shown in Figure 3.4.2.2.

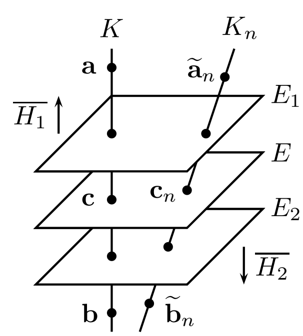


Figure 3.4.2.2

Let $\overline{H_i}$ $(i = 1, 2)$ be the closed half-space determined by E_i which does not contain E. Applying Lemmas 3.4.8 and 3.4.9 to E_i $(i = 1, 2)$, for sufficiently large n we have that $K_n \cap E_i$ is a single point, and the sequence of such points converges to $K \cap E_i$. As $K \cap E_i \in U$, for sufficiently large n we thus have that $K_n \cap E_i \in U \subseteq \mathbf{B}$. Thus, by removing finitely many n, we may assume that K_n

intersects $\mathbf{P}$ in $\overline{H_1}$ and in $\overline{H_2}$, so for each n we may let

$$\widetilde{\mathbf{a}}_n = K_n \cap \mathbf{P} \cap \overline{H_1},$$
$$\widetilde{\mathbf{b}}_n = K_n \cap \mathbf{P} \cap \overline{H_2}.$$

Note that $\{\widetilde{\mathbf{a}}_n, \widetilde{\mathbf{b}}_n\} = \{\mathbf{a}_n, \mathbf{b}_n\}$ for each n. Now, since $K_n \cap E_1$ and $K_n \cap E_2$ lie on opposite sides of $\mathbf{c}_n$ in $]\mathbf{a}_n, \mathbf{b}_n[$, we have that $K_n \cap E_1 \in]\mathbf{c}_n, \widetilde{\mathbf{a}}_n[$ and $K_n \cap E_2 \in]\mathbf{c}_n, \widetilde{\mathbf{b}}_n[$ for all n. Noting that $K \cap E_1 \subseteq]\mathbf{a}, \mathbf{c}[$ and $\mathbf{a}$ does not lie in $](K \cap E_1), \mathbf{c}[\subseteq \mathbf{B}$, we have that $\mathbf{a}$ lies in the $\overline{H_1}$. Similarly, we have $\mathbf{b} \in \overline{H_2}$. Thus $\mathbf{a}$ and $\widetilde{\mathbf{a}}_n$ lie in $\overline{H_1}$, and $\mathbf{b}$ and $\widetilde{\mathbf{b}}_n$ lie in $\overline{H_2}$, for all n.

Finally, we show that any subsequence of $(\widetilde{\mathbf{a}}_n)$ and any subsequence of $(\widetilde{\mathbf{b}}_n)$ converges to $\mathbf{a}$ and $\mathbf{b}$, respectively.

Since $\widetilde{\mathbf{a}}_n \in \mathbf{P} \cap \overline{H_1}$ for all n and $\mathbf{P} \cap \overline{H_1}$ is compact, there is a subsequence $(\widetilde{\mathbf{a}}_{n_k})$ of $(\widetilde{\mathbf{a}}_n)$ converging to some point $\widetilde{\mathbf{a}} \in \mathbf{P} \cap \overline{H_1}$. Similarly we find a convergent subsequence of $(\widetilde{\mathbf{b}}_n)$, which without loss of generality we may label $(\widetilde{\mathbf{b}}_{n_k})$, converging to a point $\widetilde{\mathbf{b}} \in \mathbf{P} \cap \overline{H_2}$.

Since $\overline{H_1}$ and $\overline{H_2}$ are disjoint, $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{b}}$ are distinct and so by Lemma 3.4.10 we have

$$(\widetilde{\mathbf{a}}_{n_k}, [\widetilde{\mathbf{a}}_{n_k} - \widetilde{\mathbf{b}}_{n_k}]) \xrightarrow{k\to\infty} (\widetilde{\mathbf{a}}, [\widetilde{\mathbf{a}} - \widetilde{\mathbf{b}}]);$$

hence

$$L(\widetilde{\mathbf{a}}_{n_k}, [\widetilde{\mathbf{a}}_{n_k} - \widetilde{\mathbf{b}}_{n_k}]) \xrightarrow{k\to\infty} L(\widetilde{\mathbf{a}}, [\widetilde{\mathbf{a}} - \widetilde{\mathbf{b}}]).$$

But $L(\widetilde{\mathbf{a}}_{n_k}, [\widetilde{\mathbf{a}}_{n_k} - \widetilde{\mathbf{b}}_{n_k}]) = L(\mathbf{p}_{n_k}, [\mathbf{x}_{n_k}])$ for each k, and

$$L(\mathbf{p}_{n_k}, [\mathbf{x}_{n_k}]) \xrightarrow{k\to\infty} L(\mathbf{p}, [\mathbf{x}]) = K,$$

so $K = L(\widetilde{\mathbf{a}}, [\widetilde{\mathbf{a}} - \widetilde{\mathbf{b}}])$. In particular, this implies that $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{b}}$ lie on K. Thus

$$\widetilde{\mathbf{a}} \in K \cap \mathbf{P} \cap \overline{H_1} = \{\mathbf{a}, \mathbf{b}\} \cap \overline{H_1} = \mathbf{a} \text{ and}$$
$$\widetilde{\mathbf{b}} \in K \cap \mathbf{P} \cap \overline{H_2} = \{\mathbf{a}, \mathbf{b}\} \cap \overline{H_2} = \mathbf{b}.$$

Since $(\widetilde{\mathbf{a}}_{n_k})$ and $(\widetilde{\mathbf{b}}_{n_k})$ were arbitrary convergent subsequences of $(\widetilde{\mathbf{a}}_n)$ and $(\widetilde{\mathbf{b}}_n)$, respectively, we obtain from Theorem A.1.10 that $\widetilde{\mathbf{a}}_n \to \mathbf{a}$ and $\widetilde{\mathbf{b}}_n \to \mathbf{b}$. This completes the proof of Proposition 3.4.12. Furthermore, we have $(\widetilde{\mathbf{a}}_n, \widetilde{\mathbf{b}}_n) \to (\mathbf{a}, \mathbf{b})$ and so

$$\mu(\mathbf{p}_n, [\mathbf{x}_n]) = \{\mathbf{a}_n, \mathbf{b}_n\} = \{\widetilde{\mathbf{a}}_n, \widetilde{\mathbf{b}}_n\} \to \{\mathbf{a}, \mathbf{b}\} = \mu(\mathbf{p}, [\mathbf{x}]).$$

This shows that μ is continuous and completes the proof of Proposition 3.4.11.

□

Whence we obtain the desired result:

Theorem 3.4.13. *The intersection map* $\gamma : (\mathbf{C}^2)_* \to \widetilde{\mathbf{P}^2}$ *is continuous.*

This completes the verification that our choice of topology on the circle set of an embeddable spherical circle plane does indeed induce a topological circle plane.

We conclude this chapter by making use of Proposition 3.4.12 to show that the map taking a pair $(\mathbf{x}, \mathbf{u}) \in \mathbf{B} \times \mathbb{S}_1^2$ to the point[5] where the ray $\mathcal{R}(\mathbf{x}, \mathbf{u})$ intersects $\mathbf{P}$ is continuous.

Proposition 3.4.14. *The map*

$$\eta : \mathbf{B} \times \mathbb{S}_1^2 \to \mathbf{P} : (\mathbf{x}, \mathbf{u}) \mapsto \mathcal{R}(\mathbf{x}, \mathbf{u}) \cap \mathbf{P}$$

is continuous.

[5] This point is unique; cf. Proposition 2.3.1.

Proof. Let $\big((\mathbf{x}_n, \mathbf{u}_n)\big)$ be a sequence in $\mathbf{B} \times \mathbb{S}^2_1$ converging to a point $(\mathbf{x}, \mathbf{u}) \in \mathbf{B} \times \mathbb{S}^2_1$. Then $(\mathbf{x}_n, [\mathbf{u}_n]) \to (\mathbf{x}, [\mathbf{u}])$ in $\mathbf{B} \times \mathbb{P}_2\mathbb{R}$ and so, letting E be a plane perpendicular to the line $L(\mathbf{x}, [\mathbf{u}])$ and intersecting $L(\mathbf{x}, [\mathbf{u}])$ at $\mathbf{x}$, we may apply Proposition 3.4.12 (note that $\mathbf{B} \times \mathbb{P}_2\mathbb{R} \subseteq (\mathbb{R}^3 \times \mathbb{P}_2\mathbb{R})_*$). Relabel E_1 and E_2 so that

$$\begin{aligned} &\overline{H_1} \subseteq \{\mathbf{y} \in \mathbb{R}^3 : \mathbf{y} \cdot (\mathbf{u} - \mathbf{x}) \geq 0\} \ \text{ and} \\ &\overline{H_2} \subseteq \{\mathbf{y} \in \mathbb{R}^3 : \mathbf{y} \cdot (\mathbf{u} - \mathbf{x}) \leq 0\}; \end{aligned}$$

see Figure 3.4.2.3.

Since $\mathbf{x}_n \to \mathbf{x}$ and $\mathbf{x}$ lies in the open subset $\mathbb{R}^3 \backslash \overline{H_2}$, and since $\mathbf{u}_n \to \mathbf{u}$, by removing finitely many n if necessary we may assume that $\mathbf{x}_n \in \mathbb{R}^3 \backslash \overline{H_2}$ and that $\mathbf{u}_n \cdot \mathbf{u} > 0$ for all n. Whence $\mathcal{R}(\mathbf{x}_n, \mathbf{u}_n) \subseteq \mathbb{R}^3 \backslash \overline{H_2}$ for each n.

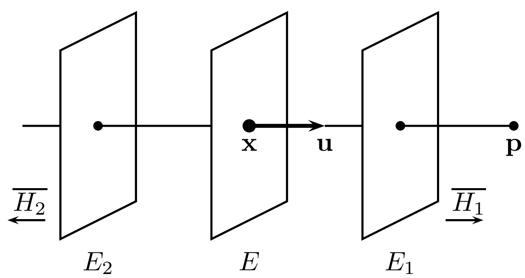


Figure 3.4.2.3

Now,

$$\mathbf{p}_n := \underbrace{\mathcal{R}(\mathbf{x}_n, \mathbf{u}_n) \cap \mathbf{P}}_{\subseteq \mathbb{R}^3 \backslash \overline{H_2}} \subseteq L(\mathbf{x}_n, [\mathbf{u}_n]) \cap \mathbf{P} = \{\underbrace{\widetilde{\mathbf{a}}_n}_{\in \overline{H_1}}, \underbrace{\widetilde{\mathbf{b}}_n}_{\in \overline{H_2}}\},$$

so $\mathbf{p}_n = \widetilde{\mathbf{a}}_n$ for each n. Similarly,

$$\mathbf{p} := \underbrace{\mathcal{R}(\mathbf{x}, \mathbf{u}) \cap \mathbf{P}}_{\subseteq \mathbb{R}^3 \backslash \overline{H_2}} \subseteq L(\mathbf{x}, [\mathbf{u}]) \cap \mathbf{P} = \{\underbrace{\mathbf{a}}_{\in \overline{H_1}}, \underbrace{\mathbf{b}}_{\in \overline{H_2}}\},$$

so $\mathbf{p} = \mathbf{a}$.

Thus $\mathbf{p}_n = \mathbf{a}_n \to \mathbf{a} = \mathbf{p}$; that is, $\eta(\mathbf{x}_n, \mathbf{u}_n) \to \eta(\mathbf{x}, \mathbf{u})$. This shows that η is continuous. $\square$

Chapter 4

The Circle Space of an ESCP

A spherical circle plane $(\mathcal{P}, \mathcal{C})$ that is also a Möbius plane is called a *flat Möbius plane*. The embeddable spherical circle plane with point set $\mathbb{S}_1^2$ is a flat Möbius plane (see [6], 3.1.1), called the *classical* flat Möbius plane.

The following result is due to Strambach [11].

Theorem. *The circle space of a flat Möbius plane is homeomorphic to* $\mathbb{P}_3\mathbb{R}$ *with one point deleted.*

In this chapter we show that Strambach's classification extends[1] to include the case of embeddable spherical circle planes. That is:

Theorem. *The circle space* $\mathbf{C}$ *of an embeddable spherical circle plane* $(\mathbf{P}, \mathbf{C})$ *is homeomorphic to the 3-dimensional projective space minus one point.*

[1] That this is indeed an extension will be verified at the end of this chapter.

4.1 Characterising the Circle Space

It is useful to partition the planes of $\mathbb{R}^3$ into those that contain the origin, and those that do not. Note that under our identification of an element $[\mathbf{x} \times d]$ of $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$ with the plane $E(\mathbf{x} \times d)$ in $\mathbb{R}^3$, we have

$$\mathbf{y} \in [\mathbf{x} \times d] \text{ if and only if } \mathbf{y} \cdot \mathbf{x} = d.$$

It is of use, then, to consider separately planes of the form $[\mathbf{u} \times 0]$, for $\mathbf{u} \in \mathbb{S}^2_1$, which are perpendicular to the vector $\mathbf{u}$ and contain the origin, and those of the form $[\mathbf{u} \times d]$, for $\mathbf{u} \in \mathbb{S}^2_1$ and $d > 0$, which are perpendicular to the vector $\mathbf{u}$ and contain $d\mathbf{u}$.

Again, let $(\mathbf{P}, \mathbf{C})$ be an embeddable spherical circle plane. We choose Cartesian coordinates for $\mathbb{R}^3$ so that the origin $\mathbf{0}$ is a point in the bounded component $\mathbf{B}$ of $\mathbb{R}^3\backslash\mathbf{P}$. For $\mathbf{y} \in \mathbb{R}^3$, let $H_\mathbf{y}$ denote the the particular open half-space determined by $[\mathbf{y} \times 0]$ given by

$$H_\mathbf{y} := \{\mathbf{z} \in \mathbb{R}^3 : \mathbf{y} \cdot \mathbf{z} > 0\}.$$

Let $\mathbf{u} \in \mathbb{S}^2_1$. Define the map

$$g_\mathbf{u} : \mathbf{P} \cap \overline{H_\mathbf{u}} \to \mathbb{R}_{\geq 0}$$

to be such that, for each $\mathbf{p} \in \mathbf{P} \cap \overline{H_\mathbf{u}}$, the value of $g_\mathbf{u}(\mathbf{p})$ is the length of the projection $(\mathbf{p} \cdot \mathbf{u})\mathbf{u}$ of $\mathbf{p}$ onto $\mathbf{u}$. Noting that $\mathbf{p} \cdot \mathbf{u} \geq 0$, we have

$$g_\mathbf{u}(\mathbf{p}) = |\mathbf{p} \cdot \mathbf{u}| = \mathbf{p} \cdot \mathbf{u}.$$

The map $g_\mathbf{u}$ is continuous by the continuity of the dot product. Hence, as

$\mathbf{P} \cap \overline{H_{\mathbf{u}}}$ is compact (being a closed and bounded subset of $\mathbb{R}^3$), so is $g_{\mathbf{u}}(\mathbf{P} \cap \overline{H_{\mathbf{u}}})$. Furthermore, since the open ray $\mathcal{R}(\mathbf{0}, \mathbf{u})$ intersects $\mathbf{P} \cap \overline{H_{\mathbf{u}}}$ (by Proposition 2.3.1), there is a $\mathbf{p} \in \mathbf{P} \cap \overline{H_{\mathbf{u}}}$ such that $g_{\mathbf{u}}(\mathbf{p}) > 0$. Thus $g_{\mathbf{u}}$ attains a positive maximum value on $\mathbf{P} \cap \overline{H_{\mathbf{u}}}$, so we may define the *distance map*

$$\mathfrak{f}_{\mathbf{P}} : \mathbb{S}^2_1 \to \mathbb{R}_{>0}$$

for $\mathbf{P}$ by

$$\mathfrak{f}_{\mathbf{P}}(\mathbf{u}) = \max_{\mathbf{p} \in \mathbf{P} \cap \overline{H_{\mathbf{u}}}} \mathbf{p} \cdot \mathbf{u}.$$

The distance map will enable us to describe the planes in $\mathbb{R}^3$ that intersect $\mathbf{P}$ non-trivially; these are the circles of $\mathbf{C}$.

Proposition 4.1.1. *A plane $E(\mathbf{u} \times d)$, with $\mathbf{u} \in \mathbb{S}^2_1$ and $d > 0$, intersects $\mathbf{P}$ at more than one point if and only if $d < \mathfrak{f}_{\mathbf{P}}(\mathbf{u})$.*

Proof. Let $\mathbf{x} = d\mathbf{u}$. Firstly, suppose that $|E(\mathbf{u} \times d) \cap \mathbf{P}| \leq 1$. By Proposition 2.3.3, $E(\mathbf{u} \times d)$ does not intersect $\mathbf{B}$; let $J_{\mathbf{x}}$ be the open half-space in $\mathbb{R}^3$ determined by $E(\mathbf{u} \times d)$ that contains $\mathbf{B}$. Then

$$\mathbf{P} \cap \overline{H_{\mathbf{x}}} \subseteq \overline{J_{\mathbf{x}}} \cap \overline{H_{\mathbf{x}}},$$

so as $\mathbf{q} \cdot \mathbf{u} \leq d$ for all $\mathbf{q} \in \overline{J_{\mathbf{x}}}$, we have

$$\mathbf{p} \cdot \mathbf{u} \leq d$$

for any $\mathbf{p} \in \mathbf{P} \cap \overline{H_{\mathbf{x}}}$ (see Figure 4.1.0.1). Thus

$$\mathfrak{f}_{\mathbf{P}}(\mathbf{u}) = \max_{\mathbf{p} \in \mathbf{P} \cap \overline{H_{\mathbf{x}}}} \mathbf{p} \cdot \mathbf{u} \leq d.$$

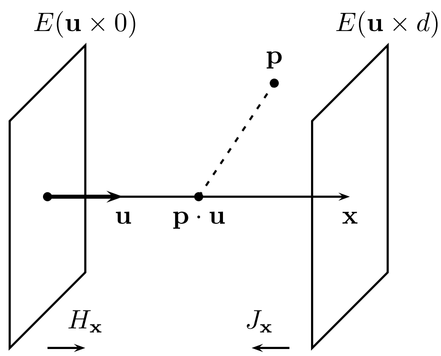


Figure 4.1.0.1

Conversely, suppose that $|E(\mathbf{u} \times d) \cap \mathbf{P}| \geq 2$; let $\mathbf{p}$ and $\mathbf{q}$ be distinct points in $E(\mathbf{u} \times d) \cap \mathbf{P}$. Let

$$\mathbf{y} \in \,]\mathbf{p}, \mathbf{q}[\, \subseteq E(\mathbf{u} \times d) \cap \mathbf{B}$$

and choose $r > 0$ so that $\mathcal{B}(\mathbf{y}, r) \subseteq \mathbf{B}$. The open ray

$$\mathcal{R}(\mathbf{y}, \mathbf{u}) = \{\mathbf{y} + k\mathbf{u} : k \in \mathbb{R}_{>0}\}$$

intersects $\mathbf{P}$ at $\mathbf{y} + k'\mathbf{u}$ for some $k' \geq r > 0$ (see Figure 4.1.0.2). Thus

$$\mathbf{y} + k'\mathbf{u} \in \mathbf{P} \cap \overline{H_{\mathbf{x}}}$$

and so, since $\mathbf{y} \in E(\mathbf{u} \times d)$ implies that $\mathbf{y} \cdot \mathbf{u} = d > 0$, we have

$$\mathfrak{f}_{\mathbf{P}}(\mathbf{u}) \geq |(\mathbf{y} + k'\mathbf{u}) \cdot \mathbf{u}| = \mathbf{y} \cdot \mathbf{u} + k' > \mathbf{y} \cdot \mathbf{u} = d.$$

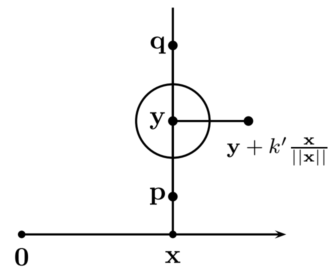


Figure 4.1.0.2 $\square$

Corollary 4.1.2. *For each* $\mathbf{u} \in \mathbb{S}^2_1$*, there is a unique point* $\mathbf{p} \in \mathbf{P} \cap \overline{H_\mathbf{u}}$ *such that* $\mathfrak{f}_\mathbf{P}(\mathbf{u}) = \mathbf{p} \cdot \mathbf{u}$*.*

Proof. Suppose there are two distinct points $\mathbf{p}$ and $\mathbf{q}$ in $\mathbf{P} \cap H_\mathbf{u}$ such that

$$\mathfrak{f}_\mathbf{P}(\mathbf{u}) = \mathbf{p} \cdot \mathbf{u} = \mathbf{q} \cdot \mathbf{u}.$$

Then $\mathbf{p}$ and $\mathbf{q}$ are distinct points of $E(\mathbf{u} \times \mathfrak{f}_\mathbf{P}(\mathbf{u})) \cap \mathbf{P}$, so by Proposition 4.1.1 we have the contradictory statement that $\mathfrak{f}_\mathbf{P}(\mathbf{u}) > \mathfrak{f}_\mathbf{P}(\mathbf{u})$. $\square$

Since the circles of $\mathbf{C}$ correspond precisely with the planes that intersect $\mathbf{P}$ non-trivially, from Propositions 2.3.2 and 4.1.1 we obtain the following characterisation of $\mathbf{C}$.

Theorem 4.1.3. *The circle space* $\mathbf{C}$ *of the embeddable spherical circle plane* $(\mathbf{P}, \mathbf{C})$ *may be written*

$$\mathbf{C} = \left\{[\mathbf{u} \times 0] : \mathbf{u} \in \mathbb{S}^2_1\right\} \cup \left\{[\mathbf{u} \times d] : 0 < d < \mathfrak{f}_\mathbf{P}(\mathbf{u}),\, \mathbf{u} \in \mathbb{S}^2_1\right\},$$

where $\mathfrak{f}_\mathbf{P} : \mathbb{S}^2_1 \to \mathbb{R}_{>0}$ *is the distance map defined for* $\mathbf{P}$*.*

4.2 Continuity of the Distance Map

Our goal is to "stretch" the space of planes in $\mathbb{R}^3$ that intersect $\mathbf{P}$ non-trivially to the space of all planes in $\mathbb{R}^3$, which is homeomorphic to $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$. In order for this mechanism to be a homeomorphism, we will need that the distance map is continuous.

Theorem 4.2.1. *The distance map for* $\mathbf{P}$

$$\mathfrak{f}_{\mathbf{P}} : \mathbb{S}^2_1 \to \mathbb{R}_{>0}, \quad \mathfrak{f}_{\mathbf{P}}(\mathbf{u}) = \max_{\mathbf{p} \in \mathbf{P} \cap \overline{H_{\mathbf{u}}}} \mathbf{p} \cdot \mathbf{u}$$

is continuous.

Proof. Write $\mathfrak{f} := \mathfrak{f}_{\mathbf{P}}$. Let $(\mathbf{u}_n)$ be a sequence in $\mathbb{S}^2_1$ converging to a point $\mathbf{u} \in \mathbb{S}^2_1$. Let $\varepsilon > 0$; we show that for sufficiently large n, $|\mathfrak{f}(\mathbf{u}_n) - \mathfrak{f}(\mathbf{u})| < \varepsilon$.

We claim there is an $r > 0$ such that for all $\mathbf{y} \in \mathcal{B}(\mathbf{u}, r) \cap \mathbb{S}^2_1$: the tangent planes to $\mathbb{S}^2_{\mathfrak{f}(\mathbf{u})+\varepsilon}$ with normal $\mathbf{y}$ do not intersect $\mathbf{P}$, but the tangent planes to $\mathbb{S}^2_{\mathfrak{f}(\mathbf{u})-\varepsilon}$ with normal $\mathbf{y}$ intersect $\mathbf{P}$ non-trivially.

Since $\overline{\mathbf{B}}$ is bounded, there is an $\widetilde{R} > 0$ such that $\overline{\mathbf{B}} \subseteq \mathcal{B}(\mathbf{0}, \widetilde{R})$. Letting $\mathbf{p} \in \mathbf{P} \cap \overline{H_{\mathbf{u}}}$ be the unique point for which $\mathfrak{f}(\mathbf{u}) = \mathbf{p} \cdot \mathbf{u}$ (cf. Corollary 4.1.2), we therefore have that $\mathbf{p}$ lies on the disc

$$D := E(\mathbf{u} \times \mathfrak{f}(\mathbf{u})) \cap \mathcal{B}(\mathfrak{f}(\mathbf{u})\mathbf{u}, R),$$

where $R = \sqrt{\widetilde{R}^2 - \mathfrak{f}(\mathbf{u})^2}$.

By radial symmetry it suffices to consider the various objects' intersections with the plane $z = 0$; choose coordinates so that $\mathbf{u} = (1, 0, 0)$. See Figure 4.2.0.3.

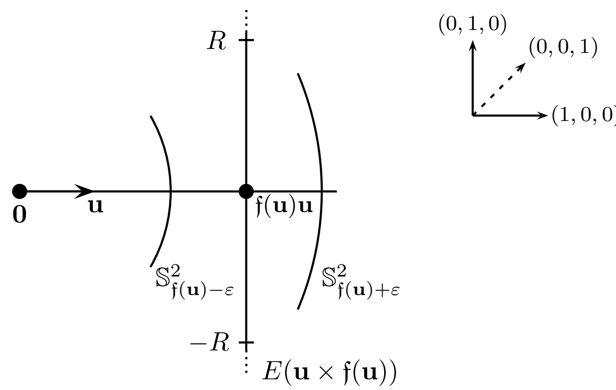


Figure 4.2.0.3

For $\theta \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$, let T_θ denote the tangent plane to $\mathbb{S}^2_{\mathfrak{f}(\mathbf{u})-\varepsilon}$ with normal $(\cos\theta, \sin\theta, 0)$. Then T_θ intersects the plane $E(\mathbf{u} \times \mathfrak{f}(\mathbf{u}))$ along the line

$$(\mathfrak{f}(\mathbf{u}), -R(\theta)) \times \mathbb{R},$$

as shown in Figure 4.2.0.4, where

$$R(\theta) = \frac{\mathfrak{f}(\mathbf{u})\cos\theta - (\mathfrak{f}(\mathbf{u}) - \varepsilon)}{\sin\theta}.$$

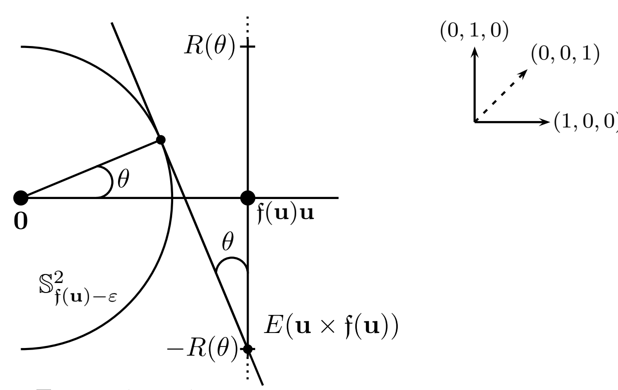


Figure 4.2.0.4

Let T_{θ_0} be the particular tangent plane for which $R(\theta_0) = R$. Note that

$$R'(\theta) = (\mathfrak{f}(\mathbf{u}) - \varepsilon)\frac{\cos\theta - \frac{\mathfrak{f}(\mathbf{u})}{\mathfrak{f}(\mathbf{u})-\varepsilon}}{\sin^2\theta} < (\mathfrak{f}(\mathbf{u}) - \varepsilon)\frac{\cos\theta - 1}{\sin^2\theta} \leq 0$$

for all $\theta \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$. Thus, for $0 < \theta < \theta_0$, the corresponding tangent planes T_θ intersect $E(\mathbf{u} \times \mathfrak{f}(\mathbf{u}))$ along the line $(\mathfrak{f}(\mathbf{u}), -R(\theta)) \times \mathbb{R}$ with $R(\theta) > R(\theta_0) = R$.

Similarly, the tangent planes T_θ with $-\theta_0 < \theta < 0$ intersect $E(\mathbf{u} \times \mathfrak{f}(\mathbf{u}))$ along the line $(\mathfrak{f}(\mathbf{u}), R(\theta)) \times \mathbb{R}$ with $R(\theta) > R$. By choosing $r_1 > 0$ so that

$$\mathcal{B}(\mathbf{u}, r_1) \cap \mathbb{S}^2_1 \subseteq \{\mathbf{y} \in \mathbb{S}^2_1 : |\mathbf{u} \cdot \mathbf{y}| > \cos\theta_0\},$$

for each $\mathbf{y} \in \mathcal{B}(\mathbf{u}, r_1) \cap \mathbb{S}^2_1$ we have that $E(\mathbf{y} \times (\mathfrak{f}(\mathbf{u}) - \varepsilon))$ does not intersect $E(\mathbf{u} \times \mathfrak{f}(\mathbf{u}))$ inside D. Thus $E(\mathbf{y} \times (\mathfrak{f}(\mathbf{u}) - \varepsilon))$ separates $\mathbf{0}$ and $\mathbf{p} \in D$, in the sense that the open line segment $]\mathbf{0}, \mathbf{p}[\subseteq \mathbf{B}$ must intersect it; hence $E(\mathbf{y} \times (\mathfrak{f}(\mathbf{u}) - \varepsilon))$ intersects $\mathbf{P}$ non-trivially by Proposition 2.3.3. Whence, by Proposition 4.1.1, we have

$$\mathfrak{f}(\mathbf{y}) > \mathfrak{f}(\mathbf{u}) - \varepsilon.$$

We now find an $r_2 > 0$ so that $\mathfrak{f}(\mathbf{y}) < \mathfrak{f}(\mathbf{u}) + \varepsilon$ for all $\mathbf{y} \in \mathcal{B}(\mathbf{u}, r_2) \cap \mathbb{S}^2_1$. First note that if

$$R \leq \sqrt{(\mathfrak{f}(\mathbf{u}) + \varepsilon)^2 - \mathfrak{f}(\mathbf{u})^2},$$

that is, $\widetilde{R} \leq \mathfrak{f}(\mathbf{u}) + \varepsilon$, then $D \subseteq \mathcal{B}(\mathbf{0}, \mathfrak{f}(\mathbf{u}) + \varepsilon)$. So, for each $\mathbf{y} \in \mathbb{S}^2_1$, the plane $E(\mathbf{y} \times (\mathfrak{f}(\mathbf{u}) + \varepsilon))$ does not intersect $\mathbf{B}$ (see Figure 4.2.0.5); hence $\mathfrak{f}(\mathbf{y}) < \mathfrak{f}(\mathbf{u}) + \varepsilon$.

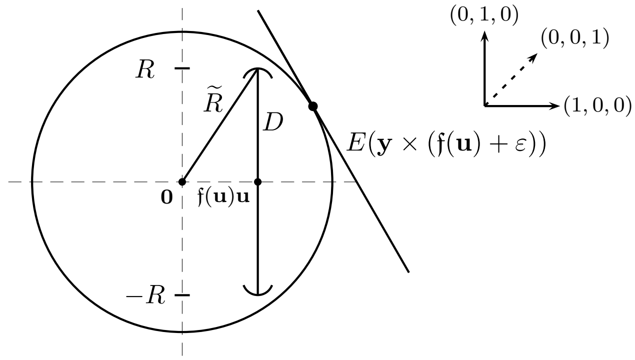


Figure 4.2.0.5

Now suppose that

$$R > \sqrt{(\mathfrak{f}(\mathbf{u}) + \varepsilon)^2 - \mathfrak{f}(\mathbf{u})^2},$$

that is, $\widetilde{R} > \mathfrak{f}(\mathbf{u}) + \varepsilon$.

For $\theta \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$, let G_θ denote the tangent plane to $\mathbb{S}^2_{\mathfrak{f}(\mathbf{u})+\varepsilon}$ with normal $(\cos\theta, \sin\theta, 0)$. Then G_θ intersects the plane $E(\mathbf{u} \times \mathfrak{f}(\mathbf{u}))$ along the line

$$(\mathfrak{f}(\mathbf{u}), Q(\theta)) \times \mathbb{R},$$

as shown in Figure 4.2.0.6, where

$$Q(\theta) = \frac{(\mathfrak{f}(\mathbf{u}) + \varepsilon) - \mathfrak{f}(\mathbf{u})\cos\theta}{\sin\theta}.$$

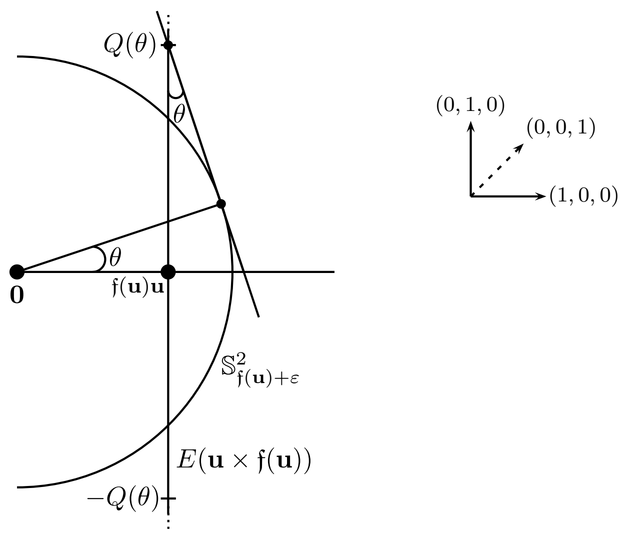


Figure 4.2.0.6

Let G_{θ_1} be the tangent plane for which $Q(\theta_1) = R$. Note that since

$$R > \sqrt{(\mathfrak{f}(\mathbf{u})+\varepsilon)^2 - \mathfrak{f}(\mathbf{u})^2},$$

we have $\theta_1 < \delta < \frac{\pi}{2}$, where

$$\cos\delta = \frac{\mathfrak{f}(\mathbf{u})}{\mathfrak{f}(\mathbf{u})+\varepsilon}.$$

Thus

$$Q'(\theta) = \frac{\mathfrak{f}(\mathbf{u}) - (\mathfrak{f}(\mathbf{u})+\varepsilon)\cos\theta}{\sin^2\theta} = (\mathfrak{f}(\mathbf{u})+\varepsilon)\frac{\cos\delta - \cos\theta}{\sin^2\theta} < 0$$

for $0 < \theta < \theta_1$. Therefore, for $0 < \theta < \theta_1$, the corresponding tangent planes G_θ intersect $E(\mathbf{u}\times\mathfrak{f}(\mathbf{u}))$ along the line $(\mathfrak{f}(\mathbf{u}), Q(\theta)) \times \mathbb{R}$ with $Q(\theta) > Q(\theta_1) = R$.

Similarly, the tangent planes G_θ with $-\theta_1 < \theta < 0$ intersect $E(\mathbf{u}\times\mathfrak{f}(\mathbf{u}))$ along the line $(\mathfrak{f}(\mathbf{u}), Q(\theta)) \times \mathbb{R}$ with $Q(\theta) > R$. Whence, by choosing $r_2 > 0$ so that

$$\mathcal{B}(\mathbf{u}, r_2) \cap \mathbb{S}_1^2 \subseteq \{\mathbf{y} \in \mathbb{S}_1^2 : |\mathbf{u} \cdot \mathbf{y}| > \cos\theta_1\},$$

for each $\mathbf{y} \in \mathcal{B}(\mathbf{u}, r_2) \cap \mathbb{S}_1^2$ we have that $E(\mathbf{y} \times (\mathfrak{f}(\mathbf{u}) + \varepsilon))$ does not intersect $E(\mathbf{u} \times \mathfrak{f}(\mathbf{u}))$ inside D. Thus $E(\mathbf{y} \times (\mathfrak{f}(\mathbf{u}) + \varepsilon))$ does not intersect $\mathbf{P} \subseteq \mathcal{B}(\mathbf{0}, \widetilde{R})$, so by Proposition 4.1.1 we have

$$\mathfrak{f}(\mathbf{y}) > \mathfrak{f}(\mathbf{u}) + \varepsilon.$$

Put $r := \min(r_1, r_2) > 0$. Since $\mathbf{u}_n \to \mathbf{u}$, there is an $N > 0$ such that $\mathbf{u}_n \in \mathcal{B}(\mathbf{u}, r) \cap \mathbb{S}_1^2$ for all $n > N$. Thus, for all $n > N$ we have that

$$\mathfrak{f}(\mathbf{u}) - \varepsilon < \mathfrak{f}(\mathbf{u}_n) < \mathfrak{f}(\mathbf{u}) + \varepsilon,$$

that is, $|\mathfrak{f}(\mathbf{u}_n) - \mathfrak{f}(\mathbf{u})| < \varepsilon$; hence $\mathfrak{f}(\mathbf{u}_n) \to \mathfrak{f}(\mathbf{u})$. This shows that $\mathfrak{f}$ is continuous at $\mathbf{u}$ and we deduce that $\mathfrak{f}$ is continuous on $\mathbb{S}_1^2$. □

In later chapters we will need to ascertain the effect "shrinking" $\mathbf{P}$ down to $\mathbb{S}_1^2$ has on the circles of $\mathbf{C}$. The map which takes each $\mathbf{u} \in \mathbb{S}_1^2$ to the unique point on $\mathbf{P}$ for which $\mathfrak{f}_{\mathbf{P}}(\mathbf{u}) = \mathbf{p} \cdot \mathbf{u}$ (cf. Corollary 4.1.2) will help us to achieve this. Let us formalise this map and show that it is continuous. Let

$$\mathfrak{h}_{\mathbf{P}} : \mathbb{S}_1^2 \to \mathbf{P}$$

be the *maximal map* for $\mathbf{P}$, defined by

$$\mathfrak{h}_{\mathbf{P}}(\mathbf{u}) = E(\mathbf{u} \times \mathfrak{f}_{\mathbf{P}}(\mathbf{u})) \cap \mathbf{P}.$$

Theorem 4.2.2. *The maximal map $\mathfrak{h}_{\mathbf{P}} : \mathbb{S}_1^2 \to \mathbf{P}$ for $\mathbf{P}$ is continuous.*

Proof. Write $\mathfrak{f} := \mathfrak{f}_{\mathbf{P}}$, $\mathfrak{h} := \mathfrak{h}_{\mathbf{P}}$. Let $(\mathbf{u}_n)$ be a sequence in $\mathbb{S}_1^2$ converging to the point $\mathbf{u} \in \mathbb{S}^2_1$. Define the sequence $(\mathbf{p}_n)$ in $\mathbf{P}$ by

$$\mathbf{p}_n = \mathfrak{h}(\mathbf{u}_n) = E(\mathbf{u}_n \times \mathfrak{f}(\mathbf{u}_n)) \cap \mathbf{P}$$

for each n, and let

$$\mathbf{p} = \mathfrak{h}(\mathbf{u}) = E(\mathbf{u} \times \mathfrak{f}(\mathbf{u})) \cap \mathbf{P}.$$

Since $\mathbf{P}$ is a compact subset of $\mathbb{R}^3$, the sequence $(\mathbf{p}_n)$ has a subsequence $(\mathbf{p}_{n_k})$ converging in $\mathbb{R}^3$ to a point $\mathbf{q} \in \mathbf{P}$. We show that $\mathbf{q} = \mathbf{p}$.

As $\mathbf{p}_{n_k} \in E(\mathbf{u}_{n_k} \times \mathfrak{f}(\mathbf{u}_{n_k}))$ for each k, we have

$$0 = \mathbf{p}_{n_k} \cdot \mathbf{u}_{n_k} - \mathfrak{f}(\mathbf{u}_{n_k}) \xrightarrow{k\to\infty} \mathbf{q} \cdot \mathbf{u} - \mathfrak{f}(\mathbf{u})$$

by the continuity of $\mathfrak{f}$ and the dot product. Hence $\mathbf{q} \cdot \mathbf{u} - \mathfrak{f}(\mathbf{u}) = 0$ and so $\mathbf{q} \in E(\mathbf{u} \times \mathfrak{f}(\mathbf{u}))$ — but $\mathbf{q} \in \mathbf{P}$, so $\mathbf{q}$ must be the unique point $E(\mathbf{u} \times \mathfrak{f}(\mathbf{u})) \cap \mathbf{P} = \mathbf{p}$. We thus have that

$$\mathfrak{h}(\mathbf{u}_n) = \mathbf{p}_n \to \mathbf{p} = \mathfrak{h}(\mathbf{u})$$

in $\mathbb{R}^3$ as $n \to \infty$, which shows that $\mathfrak{h}$ is continuous. $\square$

4.3 The Homeomorphic type of the Circle Space

In this section we construct the desired "stretching" homeomorphism from the set of planes in $\mathbb{R}^3$ intersecting $\mathbf{P}$ non-trivially, to all of $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$.

Define the mapping

$$\varphi : [0, 1) \to [0, \infty)$$

by

$$\varphi(k) = \frac{k}{1-k}.$$

Then φ has the inverse mapping given by $\varphi^{-1}(k) = \frac{k}{1+k}$. As both φ and φ^{-1} are evidently continuous, we see that φ is a homeomorphism.

Lemma 4.3.1. *The map*

$$\phi : \mathbb{R}^3\backslash\mathbf{0} \to \left\{\mathbf{x} \in \mathbb{R}^3\backslash\mathbf{0} : \tfrac{1}{||\mathbf{x}||} < \mathfrak{f}_{\mathbf{P}}\big(\tfrac{\mathbf{x}}{||\mathbf{x}||}\big)\right\}$$

given by

$$\mathbf{x} \mapsto \frac{1}{\mathfrak{f}_{\mathbf{P}}\big(\frac{\mathbf{x}}{||\mathbf{x}||}\big)\varphi^{-1}\big(\frac{1}{||\mathbf{x}||}\big)}\frac{\mathbf{x}}{||\mathbf{x}||}$$

is a homeomorphism.

Proof. Let $\mathbf{x} \in \mathbb{R}^3\backslash\mathbf{0}$; put $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$ and $d = \frac{1}{||\mathbf{x}||}$. First note that ϕ is well-defined since

$$1\Big/\Big\|\frac{1}{\mathfrak{f}_{\mathbf{P}}(\mathbf{u})\varphi^{-1}(d)}\mathbf{u}\Big\| = \mathfrak{f}_{\mathbf{P}}(\mathbf{u})\varphi^{-1}(d) < \mathfrak{f}_{\mathbf{P}}(\mathbf{u}).$$

We claim that

$$\phi^{-1} : \mathbf{x} \mapsto \frac{1}{\varphi\left(\frac{d}{\mathfrak{f}_{\mathbf{P}}(\mathbf{u})}\right)}\mathbf{u},$$

where $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$, $d = \frac{1}{||\mathbf{x}||}$, is the inverse mapping of ϕ. Let $\mathbf{x} \in \mathbb{R}^3\backslash\mathbf{0}$; put $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$ and $d = \frac{1}{||\mathbf{x}||}$. Then

$$\mathbf{x} \overset{\phi}{\longmapsto} \frac{1}{\mathfrak{f}_{\mathbf{P}}(\mathbf{u})\varphi^{-1}(d)}\mathbf{u} \overset{\phi^{-1}}{\longmapsto} \frac{1}{\varphi\left(\frac{\mathfrak{f}_{\mathbf{P}}(\mathbf{u})\varphi^{-1}(d)}{\mathfrak{f}_{\mathbf{P}}(\mathbf{u})}\right)}\mathbf{u} = \tfrac{1}{d}\mathbf{u} = \mathbf{x}.$$

Conversely, let $\mathbf{x} \in \mathbb{R}^3\backslash\mathbf{0}$ such that $d < \mathfrak{f}_{\mathbf{P}}(\mathbf{u})$, where $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$ and $d = \frac{1}{||\mathbf{x}||}$. Then

$$\mathbf{x} \overset{\phi^{-1}}{\longmapsto} \frac{1}{\varphi\left(\frac{d}{\mathfrak{f}_{\mathbf{P}}(\mathbf{u})}\right)}\mathbf{u} \overset{\phi}{\longmapsto} \frac{1}{\mathfrak{f}_{\mathbf{P}}(\mathbf{u})\varphi^{-1}\left(\varphi\left(\frac{d}{\mathfrak{f}_{\mathbf{P}}(\mathbf{u})}\right)\right)}\mathbf{u} = \tfrac{1}{d}\mathbf{u} = \mathbf{x}.$$

This verifies our claim.

Continuity of ϕ and ϕ^{-1} follows by the continuity of $\mathfrak{f}_{\mathbf{P}}$ (cf. Theorem 4.2.1),

φ, φ^{-1} and the mapping $\mathbf{x} \mapsto \frac{1}{||\mathbf{x}||}$. Thus ϕ is a homeomorphism. $\square$

Let

$$\mathcal{Y}_1 := \left\{[\mathbf{x} \times 0] : \mathbf{x} \in \mathbb{R}^3 \backslash \mathbf{0}\right\},$$

$$\mathcal{Y}_2 := \left\{[\mathbf{x} \times 1] : \mathbf{x} \in \mathbb{R}^3 \backslash \mathbf{0}\right\}.$$

Then $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are disjoint and $\mathcal{Y}_1 \cup \mathcal{Y}_2 = \mathbb{P}_3\mathbb{R} \backslash \mathbf{p_o}$. Recall that $\pi_3 : \mathbb{R}^4 \backslash \mathbf{0} \to \mathbb{P}_3\mathbb{R}$ is the canonical quotient map $\mathbf{x} \mapsto [\mathbf{x}]$. Since

$$\pi_3^{-1}(\mathcal{Y}_1) = (\mathbb{R}^3 \times 0) \cap (\mathbb{R}^4 \backslash \mathbf{0})$$

is closed in $\mathbb{R}^4 \backslash \mathbf{0}$, we see that $\mathcal{Y}_1$ is closed, and hence $\mathcal{Y}_2$ is open, in $\mathbb{P}_3\mathbb{R}$. Furthermore, it is obvious that $\mathcal{Y}_1 \approx \mathbb{P}_2\mathbb{R}$ via the map $[\mathbf{x} \times 0] \mapsto [\mathbf{x}]$.

The map $\xi : \mathcal{Y}_2 \to \mathbb{S}^2 \times \mathbb{R}_{>0}$, given by

$$\xi : [\mathbf{y} \times m] \mapsto \frac{|m|}{m} \frac{\mathbf{y}}{||\mathbf{y}||} \times \frac{|m|}{||\mathbf{y}||}$$

for $\mathbf{y} \in \mathbb{R}^3 \backslash \mathbf{0}$, $m \neq 0$, is well-defined; for $l \neq 0$:

$$[l\mathbf{y} \times lm] \mapsto \frac{|lm|}{lm} \frac{l\mathbf{y}}{||l\mathbf{y}||} \times \frac{|lm|}{||l\mathbf{y}||} = \frac{|m|}{m} \frac{\mathbf{y}}{||\mathbf{y}||} \times \frac{|m|}{||\mathbf{y}||}.$$

So, as

$$\xi \circ \pi_3 : \mathbf{y} \times m \mapsto \frac{|m|}{m} \frac{\mathbf{y}}{||\mathbf{y}||} \times \frac{|m|}{||\mathbf{y}||}$$

is continuous, ξ is continuous. The inverse mapping of ξ, $\mathbf{u} \times d \mapsto [\mathbf{u} \times d]$, is evidently also continuous. This shows that $\mathcal{Y}_2 \approx \mathbb{S}^2 \times \mathbb{R}_{>0}$.

Lemma 4.3.2. *A sequence* $\left(\frac{\mathbf{z}_n \times 1}{||\mathbf{z}_n \times 1||}\right)$ *in* $\mathbb{R}^3 \backslash \mathbf{0} \times \mathbb{R}$ *converges to a point* $\frac{\mathbf{z} \times 0}{||\mathbf{z} \times 0||}$ *if and only if* $\frac{\mathbf{z}_n}{||\mathbf{z}_n||} \to \frac{\mathbf{z}}{||\mathbf{z}||}$ *in* $\mathbb{R}^3 \backslash \mathbf{0}$ *and* $\frac{1}{||\mathbf{z}_n||} \to 0$ *in* $\mathbb{R}$.

Proof. Suppose that $\frac{\mathbf{z}_n \times 1}{||\mathbf{z}_n \times 1||} \to \frac{\mathbf{z} \times 0}{||\mathbf{z} \times 0||}$; then $\frac{\mathbf{z}_n}{||\mathbf{z}_n \times 1||} \to \frac{\mathbf{z}}{||\mathbf{z}||}$ and $\frac{1}{||\mathbf{z}_n \times 1||} \to 0$. The latter implies that

$$\frac{1}{\sqrt{||\mathbf{z}_n||^2+1}} \to 0,$$

so $||\mathbf{z}_n|| \to \infty$ — otherwise, if there is an $A > 0$ such that $0 < ||\mathbf{z}_n|| < A$ for all n, then

$$\frac{1}{\sqrt{||\mathbf{z}_n||^2+1}} > \frac{1}{A^2+1} > 0$$

for all n. Hence $\frac{1}{||\mathbf{z}_n||} \to 0$ and, in particular,

$$\frac{||\mathbf{z}_n||}{||\mathbf{z}_n \times 1||} = \frac{1}{\sqrt{1+\frac{1}{||\mathbf{z}_n||^2}}} \to 1.$$

We thus obtain that

$$\begin{aligned}
\left\| \frac{\mathbf{z}_n}{||\mathbf{z}_n||} - \frac{\mathbf{z}}{||\mathbf{z}||} \right\| &= \left\| \frac{\mathbf{z}_n}{||\mathbf{z}_n||} - \frac{\mathbf{z}_n}{||\mathbf{z}_n \times 1||} \right\| + \left\| \frac{\mathbf{z}_n}{||\mathbf{z}_n \times 1||} - \frac{\mathbf{z}}{||\mathbf{z}||} \right\| \\
&= \left\| \frac{\mathbf{z}_n}{||\mathbf{z}_n||} - \frac{||\mathbf{z}_n||}{||\mathbf{z}_n \times 1||} \frac{\mathbf{z}_n}{||\mathbf{z}_n||} \right\| + \left\| \frac{\mathbf{z}_n}{||\mathbf{z}_n \times 1||} - \frac{\mathbf{z}}{||\mathbf{z}||} \right\| \\
&= \left| 1 - \frac{||\mathbf{z}_n||}{||\mathbf{z}_n \times 1||} \right| + \left\| \frac{\mathbf{z}_n}{||\mathbf{z}_n \times 1||} - \frac{\mathbf{z}}{||\mathbf{z}||} \right\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$.

Conversely, suppose that $\frac{\mathbf{z}_n}{||\mathbf{z}_n||} \to \frac{\mathbf{z}}{||\mathbf{z}||}$ and $\frac{1}{||\mathbf{z}_n||} \to 0$. Then

$$\begin{aligned}
\frac{\mathbf{z}_n \times 1}{||\mathbf{z}_n \times 1||} &= \frac{\mathbf{z}_n}{||\mathbf{z}_n \times 1||} \times \frac{1}{||\mathbf{z}_n \times 1||} \\
&= \frac{1}{\sqrt{1+\frac{1}{||\mathbf{z}_n||^2}}} \frac{\mathbf{z}_n}{||\mathbf{z}_n||} \times \frac{\frac{1}{||\mathbf{z}_n||}}{\sqrt{1+\frac{1}{||\mathbf{z}_n||^2}}} \\
&\to \frac{\mathbf{z}}{||\mathbf{z}||} \times 0 = \frac{\mathbf{z} \times 0}{||\mathbf{z} \times 0||}
\end{aligned}$$

as $n \to \infty$. □

Lemma 4.3.3. *If the map $\tau : \mathbb{R}^3\backslash\mathbf{0} \to \tau(\mathbb{R}^3\backslash\mathbf{0}) \subseteq \mathbb{R}^3\backslash\mathbf{0}$ is a homeomorphism, then so is the map $\mathcal{G} : \mathcal{Y}_2 \to \mathcal{G}(\mathcal{Y}_2) \subseteq \mathcal{Y}_2$, which is given by*

$$\mathcal{G} : [\mathbf{x} \times m] \mapsto \left[\tau(\tfrac{\mathbf{x}}{m}) \times 1\right],$$

for $\mathbf{x} \in \mathbb{R}^3\backslash\mathbf{0}$, $m \neq 0$.

Proof. Note that $\mathcal{G}$ is well-defined since, for $l \neq 0$:

$$\mathcal{G}([l\mathbf{x} \times lm]) = [\tau(\tfrac{l\mathbf{x}}{lm}) \times 1] = [\tau(\tfrac{\mathbf{x}}{m}) \times 1] = \mathcal{G}([\mathbf{x} \times m]).$$

Similarly, the mapping $[\mathbf{x}\times m] \mapsto [\tau^{-1}(\frac{\mathbf{x}}{m})\times 1]$ is well-defined and readily seen to be the inverse of $\mathcal{G}$. Since the mapping $\nu : \mathbf{x}\times 1 \mapsto \tau(\mathbf{x})\times 1$ is a homeomorphism (with inverse $\mathbf{x} \times 1 \mapsto \tau^{-1}(\mathbf{x}) \times 1$), it follows from the commutative diagram

$$\begin{tikzcd}
\mathbf{x} \times 1 \arrow[r, mapsto, "\nu"] \arrow[d, mapsto, "\pi_3"'] & \tau(\mathbf{x}) \times 1 \arrow[d, mapsto, "\pi_3"] \\
{[\mathbf{x} \times 1]} \arrow[r, mapsto, "\mathcal{G}"'] & {[\tau(\mathbf{x}) \times 1]}
\end{tikzcd}$$

and Theorem A.1.2 that $\mathcal{G}$ is also a homeomorphism. ☐

Lemma 4.3.4. *Let $\tau : \mathbb{R}^3\backslash\mathbf{0} \to \tau(\mathbb{R}^3\backslash\mathbf{0}) \subseteq \mathbb{R}^3\backslash\mathbf{0}$ be a homeomorphism such that*

$$\frac{\tau(\mathbf{x})}{||\tau(\mathbf{x})||} = \frac{\mathbf{x}}{||\mathbf{x}||}$$

for every $\mathbf{x} \in \mathbb{R}^3\backslash\mathbf{0}$. Let $([\mathbf{x}_n \times 0])$, $([\mathbf{y}_n \times 1])$ be sequences in $\mathcal{Y}_1$ and $\mathcal{Y}_2$, respectively. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3\backslash\mathbf{0}$.

(i) If $[\mathbf{x}_n \times 0] \to [\mathbf{x} \times 0]$, then $[\tau(\mathbf{x}_n) \times 0] \to [\tau(\mathbf{x}) \times 0]$;

(ii) if $[\mathbf{y}_n \times 1] \to [\mathbf{y} \times 1]$, then $[\tau(\mathbf{y}_n) \times 1] \to [\tau(\mathbf{y}) \times 1]$.

(iii) Suppose that τ also has the property that $\frac{1}{||\tau(\mathbf{z}_n)||} \to 0$ in $\mathbb{R}$ for every sequence $(\mathbf{z}_n)$ in $\mathbb{R}^3\backslash\mathbf{0}$ such that $\frac{1}{||\mathbf{z}_n||} \to 0$. If $[\mathbf{y}_n \times 1] \to [\mathbf{y} \times 0]$, then

$[\tau(\mathbf{y}_n) \times 1] \to [\tau(\mathbf{y}) \times 0]$.

Proof. (*i*) Let $(\mathbf{u}_n)$ be the sequence given by $\mathbf{u}_n = \frac{\mathbf{x}_n}{||\mathbf{x}_n||}$ for each n. Then $\frac{\mathbf{x}_n \times 0}{||\mathbf{x}_n \times 0||} = \mathbf{u}_n \times 0$ for each n and $\frac{\mathbf{x} \times 0}{||\mathbf{x} \times 0||} = \mathbf{u} \times 0$, so

$$\varrho_3([\mathbf{x}_n \times 0], [\mathbf{x} \times 0]) = \min\{||\mathbf{u}_n - \mathbf{u}||, ||\mathbf{u}_n + \mathbf{u}||\}.$$

Similarly, $\frac{\tau(\mathbf{x}_n) \times 0}{||\tau(\mathbf{x}_n) \times 0||} = \frac{\tau(\mathbf{x}_n)}{||\tau(\mathbf{x}_n)||} \times 0$ and $\frac{\tau(\mathbf{x}) \times 0}{||\tau(\mathbf{x}) \times 0||} = \frac{\tau(\mathbf{x})}{||\tau(\mathbf{x})||} \times 0$. Hence, since $\frac{\tau(\mathbf{x}_n)}{||\tau(\mathbf{x}_n)||} = \mathbf{u}_n$ and $\frac{\tau(\mathbf{x})}{||\tau(\mathbf{x})||} = \mathbf{u}$, we have that

$$\begin{aligned} \varrho_3([\tau(\mathbf{x}_n) \times 0], [\tau(\mathbf{x}) \times 0]) &= \min\{||\mathbf{u}_n - \mathbf{u}||, ||\mathbf{u}_n + \mathbf{u}||\} \\ &= \varrho_3([\mathbf{x}_n \times 0], [\mathbf{x} \times 0]) \to 0 \end{aligned}$$

as $n \to \infty$.

(*ii*) This follows from the continuity of the map $\xi : [\mathbf{y} \times 1] \mapsto [\tau(\mathbf{y}) \times 1]$ (cf. Lemma 4.3.3).

(*iii*) Suppose firstly that $\frac{\mathbf{y}_n \times 1}{||\mathbf{y}_n \times 1||} \to \frac{\mathbf{y} \times 0}{||\mathbf{y} \times 0||}$. By Lemma 4.3.2, we have that $\frac{\mathbf{y}_n}{||\mathbf{y}_n||} \to \frac{\mathbf{y}}{||\mathbf{y}||}$ and $\frac{1}{||\mathbf{y}_n||} \to 0$. Applying the hypotheses on τ, we therefore have

$$\frac{\tau(\mathbf{y}_n)}{||\tau(\mathbf{y}_n)||} = \frac{\mathbf{y}_n}{||\mathbf{y}_n||} \to \frac{\mathbf{y}}{||\mathbf{y}||}$$

and $\frac{1}{||\tau(\mathbf{y}_n)||} \to 0$, respectively. Hence, by Lemma 4.3.2, $\frac{\tau(\mathbf{y}_n) \times 1}{||\tau(\mathbf{y}_n) \times 1||} \to \frac{\mathbf{y} \times 0}{||\mathbf{y} \times 0||}$, so $[\tau(\mathbf{y}_n) \times 1] \to [\mathbf{y} \times 0]$.

On the other hand, suppose there are subsequences $\left(\frac{\mathbf{y}_m \times 1}{||\mathbf{y}_m \times 1||}\right)$, $\left(\frac{\mathbf{y}_{m'} \times 1}{||\mathbf{y}_{m'} \times 1||}\right)$ of $\left(\frac{\mathbf{y}_n \times 1}{||\mathbf{y}_n \times 1||}\right)$, where N_1 and N_2 partition $\mathbb{N}$, such that

$$\left(\frac{\mathbf{y}_m \times 1}{||\mathbf{y}_m \times 1||}\right) \xrightarrow{m \to \infty} \frac{\mathbf{y} \times 0}{||\mathbf{y} \times 0||}$$

and

$$\left(\frac{\mathbf{y}_{m'} \times 1}{||\mathbf{y}_{m'} \times 1||}\right) \xrightarrow{m' \to \infty} -\frac{\mathbf{y} \times 0}{||\mathbf{y} \times 0||}.$$

By similar arguments to those above we obtain that $\frac{1}{||\mathbf{y}_m||} \to 0$, $\frac{1}{||\mathbf{y}_{m'}||} \to 0$ and so $\frac{1}{||\tau(\mathbf{y}_m)||} \to 0$, $\frac{1}{||\tau(\mathbf{y}_{m'})||} \to 0$; also,

$$\frac{\tau(\mathbf{y}_m)}{||\tau(\mathbf{y}_m)||} = \frac{\mathbf{y}_m}{||\mathbf{y}_m||} \xrightarrow{m \to \infty} \frac{\mathbf{y}}{||\mathbf{y}||},$$
$$\frac{\tau(\mathbf{y}_{m'})}{||\tau(\mathbf{y}_{m'})||} = \frac{\mathbf{y}_{m'}}{||\mathbf{y}_{m'}||} \xrightarrow{m' \to \infty} -\frac{\mathbf{y}}{||\mathbf{y}||}.$$

By Lemma 4.3.2, we therefore have that

$$\frac{\tau(\mathbf{y}_m) \times 1}{||\tau(\mathbf{y}_m) \times 1||} \xrightarrow{m \to \infty} \frac{\mathbf{y} \times 0}{||\mathbf{y} \times 0||} \text{ and } \frac{\tau(\mathbf{y}_{m'}) \times 1}{||\tau(\mathbf{y}_{m'}) \times 1||} \xrightarrow{m' \to \infty} \frac{\mathbf{y} \times 0}{||\mathbf{y} \times 0||}.$$

That is, by Proposition 2.2.1, $[\tau(\mathbf{y}_n) \times 1] \to [\mathbf{y} \times 0]$. □

The following map shall be of use throughout this thesis when we consider "stretching" the set of planes in $\mathbb{R}^3$.

Theorem 4.3.5. *Let* $\tau : \mathbb{R}^3\backslash\mathbf{0} \to \tau(\mathbb{R}^3\backslash\mathbf{0}) \subseteq \mathbb{R}^3\backslash\mathbf{0}$ *be a homeomorphism such that*

$$\frac{\tau(\mathbf{x})}{||\tau(\mathbf{x})||} = \frac{\mathbf{x}}{||\mathbf{x}||} = \frac{\tau^{-1}(\mathbf{x})}{||\tau^{-1}(\mathbf{x})||}$$

for every $\mathbf{x} \in \mathbb{R}^3\backslash\mathbf{0}$. *Furthermore, suppose that for any sequence* $(\mathbf{z}_n)$ *in* $\mathbb{R}^3\backslash\mathbf{0}$ *such that* $\frac{1}{||\mathbf{z}_n||} \to 0$, *then* $\frac{1}{||\tau(\mathbf{z}_n)||} \to 0$ *and* $\frac{1}{||\tau^{-1}(\mathbf{z}_n)||} \to 0$ *in* $\mathbb{R}$.

Then the mapping $\mathcal{T} : \mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o} \to \mathcal{T}(\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}) \subseteq \mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$, *defined on the disjoint subsets* $\mathcal{Y}_1$ *and* $\mathcal{Y}_2$ *by*

$$[\mathbf{x} \times 0] \mapsto [\mathbf{x} \times 0],$$
$$[\mathbf{x} \times 1] \mapsto [\tau(\mathbf{x}) \times 1],$$

respectively, is a homeomorphism.

Proof. Let (Y_n) be a sequence in $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$ converging to Y. If $Y \in \mathcal{Y}_2$, since $\mathcal{Y}_2$ is open we have that $Y_n \in \mathcal{Y}_2$ for sufficiently large n. Writing $Y_n = [\mathbf{x}_n \times 1]$ for each n, we have $[\mathbf{x}_n \times 1] \to [\mathbf{x} \times 1]$, and so by Lemma 4.3.4($ii$):

$$\mathcal{T}(Y_n) = [\tau(\mathbf{x}_n) \times 1] \to [\tau(\mathbf{x}) \times 1] = \mathcal{T}(Y).$$

Now suppose that $Y \in \mathcal{Y}_1$; put $Y = [\mathbf{x} \times 0]$. If $Y_n \in \mathcal{Y}_1$ for only finitely many n, by removing these elements from the sequence and relabelling we have $Y_n \in \mathcal{Y}_2$ for all n. Writing $Y_n = [\mathbf{x}_n \times 1]$ for each n, we have $[\mathbf{x}_n \times 1] \to [\mathbf{x} \times 0]$ and so by Lemma 4.3.4(iii):

$$\mathcal{T}(Y_n) = [\tau(\mathbf{x}_n) \times 1] \to [\tau(\mathbf{x}) \times 0] = \mathcal{T}(Y).$$

Suppose now that $Y_n \in \mathcal{Y}_1$ for infinitely many n. If there is an $N > 0$ such that $Y_n \in \mathcal{Y}_1$ for all $n > N$, putting $Y_n = [\mathbf{x}_n \times 0]$, we obtain from Lemma 4.3.4(i) that

$$\mathcal{T}(Y_n) = [\tau(\mathbf{x}_n) \times 0] \to [\tau(\mathbf{x}) \times 0] = \mathcal{T}(Y).$$

On the other hand, let $(Y_m)_{m \in N_1}$ and $(Y_{m'})_{m' \in N_2}$ be infinite subsequences of (Y_n) corresponding to the elements in $\mathcal{Y}_1$ and $\mathcal{Y}_2$, respectively, where N_1 and N_2 partition $\mathbb{N}$. Let $Y_m = [\mathbf{x}_m \times 0]$, $Y_{m'} = [\mathbf{x}_{m'} \times 1]$ for each $m \in N_1$, $m' \in N_2$. By parts (i) and (iii) of Lemma 4.3.4, respectively, we have

$$\mathcal{T}(Y_m) = [\tau(\mathbf{x}_m) \times 0] \to [\tau(\mathbf{x}) \times 0] = \mathcal{T}(Y)$$

and

$$\mathcal{T}(Y_{m'}) = [\tau(\mathbf{x}_{m'}) \times 1] \to [\tau(\mathbf{x}) \times 0] = \mathcal{T}(Y).$$

Let $\varepsilon > 0$. By the above, there are $K, L > 0$ such that $\varrho_3(\mathcal{T}(Y_m), \mathcal{T}(Y)) < \varepsilon$ and $\varrho_3(\mathcal{T}(Y_{m'}), \mathcal{T}(Y)) < \varepsilon$ for all $m > K$, $m' > L$. For each $n > \max\{K, L\}$, then, we have

$$\varrho_3(\mathcal{T}(Y_n), \mathcal{T}(Y)) = \begin{cases} \varrho_3(\mathcal{T}(Y_m), \mathcal{T}(Y)) & \text{if } n \in \{m \in N_1 : m > K\}, \text{ or} \\ \varrho_3(\mathcal{T}(Y_{m'}), \mathcal{T}(Y)) & \text{if } n \in \{m' \in N_2 : m' > L\} \end{cases}$$
$$< \varepsilon,$$

so $\mathcal{T}(Y_n) \to \mathcal{T}(Y)$. This shows that $\mathcal{T}$ is continuous.

Replacing τ with τ^{-1} in the definition of $\mathcal{T}$ defines $\mathcal{T}^{-1}$, and replacing τ with τ^{-1} in the above proof shows that $\mathcal{T}^{-1}$ is continuous. Whence $\mathcal{T}$ is a homeomorphism. $\square$

We are now in a position to give a homeomorphism from $\mathbf{C}$ to $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$.

Proposition 4.3.6. *The map*

$$\widetilde{\Phi} : \mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o} \to \big\{[\mathbf{x} \times 0] : \mathbf{x} \in \mathbb{R}^3\backslash\mathbf{0}\big\} \cup \big\{[\mathbf{x} \times 1] : \mathbf{x} \in \mathbb{R}^3\backslash\mathbf{0}, \tfrac{1}{||\mathbf{x}||} < \mathfrak{f}_{\mathbf{P}}\big(\tfrac{\mathbf{x}}{||\mathbf{x}||}\big)\big\}$$

defined on the disjoint subsets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ by

$$\mathrm{id}|_{\mathcal{Y}_1} : [\mathbf{x} \times 0] \mapsto [\mathbf{x} \times 0],$$
$$\Phi : [\mathbf{x} \times 1] \mapsto [\phi(\mathbf{x}) \times 1],$$

respectively, is a homeomorphism.

Proof. Note that if $(\mathbf{x}_n)$ is a sequence in $\mathbb{R}^3\backslash\mathbf{0}$ such that $d_n := \frac{1}{||\mathbf{x}_n||} \to 0$, putting $\mathbf{u}_n := \frac{\mathbf{x}_n}{||\mathbf{x}_n||}$ we have

$$\frac{1}{||\phi(\mathbf{x}_n)||} = \mathfrak{f}_{\mathbf{P}}(\mathbf{u}_n)\varphi^{-1}(d_n) \to 0$$

and

$$\frac{1}{||\phi^{-1}(\mathbf{x}_n)||} = \varphi\left(\frac{d_n}{\mathfrak{f}_{\mathbf{P}}(\mathbf{u}_n)}\right) \to 0$$

as $n \to 0$. So in the statement of Theorem 4.3.5 we may take $\tau := \phi$; then $\mathcal{T} = \widetilde{\Phi}$ is a homeomorphism, as required. □

Letting $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$ and $d = \frac{1}{||\mathbf{x}||}$ in the statement of Proposition 4.3.6, we obtain the desired classification of $\mathbf{C}$:

Theorem 4.3.7. *The circle space*

$$\mathbf{C} = \left\{[\mathbf{u} \times 0] : \mathbf{u} \in \mathbb{S}_1^2\right\} \cup \left\{[\mathbf{u} \times d] : 0 < d < \mathfrak{f}_{\mathbf{P}}(\mathbf{u}),\ \mathbf{u} \in \mathbb{S}_1^2\right\}$$

of the embeddable circle plane $(\mathbf{P}, \mathbf{C})$ *is homeomorphic to* $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$.

4.4 An ESCP that is not a Flat Möbius Plane

We conclude this chapter by constructing an embeddable spherical circle plane $(\mathbf{Q}, \mathbf{G})$ that is not a flat Möbius plane, and thus confirm that our classification of the circle space is indeed a new result.

We give a point space $\mathbf{Q} \subseteq \mathbb{R}^3$ homeomorphic to $\mathbb{S}_1^2$ that is strictly convex and find two distinct points $\mathbf{p}$, $\mathbf{q}$, a plane F (whose intersection with $\mathbf{Q}$ is a circle of $\mathbf{G}$) through $\mathbf{q}$, and *two different* planes through $\mathbf{p}$ whose respective intersections with $\mathbf{Q}$ intersect $\mathbf{Q} \cap F$ at $\mathbf{q}$ only. Doing so violates the axiom of touching.

Fix $r > 0$ and define the space $\mathbf{Q} = \mathbf{Q}(r)$ to be the union, $\mathbf{Q}_1(r) \cup \mathbf{Q}_2(r)$, where

$$\mathbf{Q}_1(r) := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ x \geq 0\},$$
$$\mathbf{Q}_2(r) := \{(x, y, z) \in \mathbb{R}^3 : (x - r)^2 + y^2 + z^2 = r^2 + 1, x \leq 0\}.$$

That is, $\mathbf{Q}$ is made up of a hemisphere and a fitting cap of a larger sphere; see Figure 4.4.0.7.

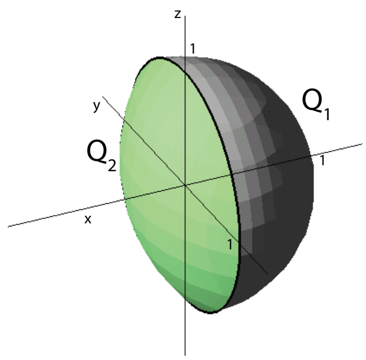


Figure 4.4.0.7: $\mathbf{Q}$ is the union of two portions of spheres, $\mathbf{Q}_1 = \mathbf{Q}(r)$ and $\mathbf{Q}_2 = \mathbf{Q}_2(r)$.

Let us first show that $\mathbf{Q}$ is homeomorphic to $\mathbb{S}_1^2$ and is strictly convex, so that $(\mathbf{Q}, \mathbf{G})$ is indeed a particular case of an embeddable spherical circle plane.

Lemma 4.4.1. *The map $h : \mathbf{Q} \to \mathbb{S}_1^2$ defined by*

$$h : (x, y, z) \mapsto \begin{cases} (x, y, z) & x \geq 0; \\ (-\sqrt{1 - y^2 - z^2}, y, z) & x \leq 0, \end{cases}$$

is a homeomorphism.

Proof. We claim the inverse mapping $h^{-1} : \mathbb{S}_1^2 \to \mathbf{Q}$ is given by

$$h^{-1} : (x, y, z) \mapsto \begin{cases} (x, y, z) & x \geq 0; \\ (r - \sqrt{r^2 + 1 - y^2 - z^2}, y, z) & x \leq 0. \end{cases}$$

Note that h is well-defined since, if $(0, y, z) \in \mathbf{Q}$, then $y^2 + z^2 = 1$ and so $-\sqrt{1 - y^2 - z^2} = 0$. Similarly, if $(0, y, z) \in \mathbb{S}_1^2$ then $y^2 + z^2 = 1$ and so $r - \sqrt{r^2 + 1 - y^2 - z^2} = 0$; hence h^{-1} is also well-defined. Let us verify that $h \circ h^{-1} = \mathrm{id}_{\mathbf{Q}}$ and $h^{-1} \circ h = \mathrm{id}_{\mathbb{S}_1^2}$.

For $(x, y, z) \in \mathbf{Q}$, noting that if $x \leq 0$ then $(x - r)^2 + y^2 + z^2 = r^2 + 1$ and so $x = r - \sqrt{r^2 + 1 - y^2 - z^2}$, we have

$$(x, y, z) \overset{h}{\mapsto} \begin{cases} (x, y, z) & x \geq 0; \\ (-\sqrt{1 - y^2 - z^2}, y, z) & x \leq 0, \end{cases}$$
$$\xmapsto{h^{-1}} \begin{cases} (x, y, z) & x \geq 0; \\ (r - \sqrt{r^2 + 1 - y^2 - z^2}, y, z) = (x, y, z) & x \leq 0. \end{cases}$$

Similarly, for $(x, y, z) \in \mathbb{S}_1^2$, we have

$$(x, y, z) \xmapsto{h^{-1}} \begin{cases} (x, y, z) & x \geq 0; \\ (r - \sqrt{r^2 + 1 - y^2 - z^2}, y, z) & x \leq 0, \end{cases}$$
$$\xmapsto{h^{-1}} \begin{cases} (x, y, z) & x \geq 0; \\ (-\sqrt{1 - y^2 - z^2}, y, z) = (x, y, z) & x \leq 0. \end{cases}$$

Define closed half-spaces $H_{\geq 0}$, $H_{\leq 0}$ in $\mathbb{R}^3$ by

$$H_{\geq 0} := \{(x, y, z) \in \mathbb{R}^3 : x \geq 0\}$$

and

$$H_{\leq 0} := \{(x, y, z) \in \mathbb{R}^3 : x \leq 0\},$$

respectively. Since h is continuous on each of the closed subsets $\mathbf{Q} \cap H_{\geq 0}$ and $\mathbf{Q} \cap H_{\leq 0}$ of $\mathbf{Q}$, and since h^{-1} is continuous on each of the closed subsets $\mathbb{S}_1^2 \cap H_{\geq 0}$ and $\mathbb{S}_1^2 \cap H_{\leq 0}$ of $\mathbb{S}_1^2$, we obtain from Theorem A.1.8 that h and h^{-1} are continuous on $\mathbf{Q}$ and $\mathbb{S}_1^2$, respectively. Thus $h : \mathbf{Q} \to \mathbb{S}_1^2$ is a homeomorphism. □

Lemma 4.4.2. *The point space $\mathbf{Q} = \mathbf{Q}_1 \cup \mathbf{Q}_2$ is strictly convex.*

Proof. Let $F = [(1, 0, 0)]^{\perp}$. Denote the line in $\mathbb{R}^3$ passing through two distinct points $\mathbf{p}$, $\mathbf{q}$ by $M_{\mathbf{p},\mathbf{q}}$. That is,

$$M_{\mathbf{p},\mathbf{q}} := L(\mathbf{p}, [\mathbf{p} - \mathbf{q}]).$$

We wish to show that if $\mathbf{p}$, $\mathbf{q}$ are two distinct points in $\mathbf{Q}$, then the line $M_{\mathbf{p},\mathbf{q}}$ intersects $\mathbf{Q}$ at no other point.

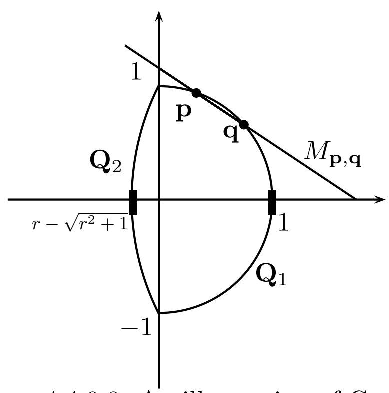
• Case 1: Suppose that $\mathbf{p}$, $\mathbf{q} \in \mathbf{Q}_1$. Since $\mathbf{Q} \subseteq \overline{\mathcal{B}(\mathbf{0},1)}$ and because $H_{\geq 0}$ is strictly convex, we have

$$M_{\mathbf{p},\mathbf{q}} \cap \mathbf{Q} \subseteq M_{\mathbf{p},\mathbf{q}} \cap \overline{\mathcal{B}(\mathbf{0},1)} =]\mathbf{p},\mathbf{q}[\subseteq H_{\geq 0}.$$

Therefore

$$M_{\mathbf{p},\mathbf{q}} \cap \mathbf{Q} \subseteq H_{\geq 0} \cap \mathbf{Q} = \mathbf{Q}_1$$

and the result follows from the strict convexity of $\mathbb{S}^2_1$.



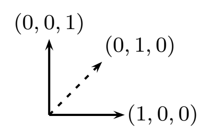


Figure 4.4.0.8: An illustration of Case 1.

• Case 2: Suppose that $\mathbf{p}$, $\mathbf{q} \in \mathbf{Q}_2$. This is analogous to the previous case because, similarly to $\mathbb{S}^2_1$, the space $\mathbb{S}^2_{\sqrt{r^2+1}}$ is strictly convex.

• Case 3: Now suppose that $\mathbf{p} \in \mathbf{Q}_1$ and $\mathbf{q} \in \mathbf{Q}_2$. If $M_{\mathbf{p},\mathbf{q}} \cap H_{\geq 0}$ intersects $\mathbf{Q}_1$ at a point other than $\mathbf{p}$, then because of the strict convexity of $\mathbb{S}^2_1$, we must have that $M_{\mathbf{p},\mathbf{q}}$ intersects F outside the unit disc on F,

$$\mathbf{D}^2 := \{(0,y,z) : y^2+z^2 < 1\}.$$

But then $M_{\mathbf{p},\mathbf{q}}$ cannot intersect $\mathbf{Q}_2$ — a contradiction. Hence $M_{\mathbf{p},\mathbf{q}}$ intersects

$\mathbf{Q}$ at $\mathbf{p}$ only. Similarly $M_{\mathbf{p},\mathbf{q}} \cap H_{\leq 0}$ intersects $\mathbf{Q}_2$ at $\mathbf{q}$ only. Thus $M_{\mathbf{p},\mathbf{q}} \cap \mathbf{Q} = \{\mathbf{p}, \mathbf{q}\}$. $\square$

From Lemmas 4.4.1 and 4.4.2 we may therefore define the embeddable spherical circle plane $(\mathbf{Q}, \mathbf{G})$ on $\mathbf{Q}$.

We conclude by showing that $(\mathbf{Q}, \mathbf{G})$ is not a Möbius plane.

Theorem 4.4.3. *The embeddable spherical circle plane* $(\mathbf{Q}, \mathbf{G})$ *is not a Möbius plane.*

Proof. Let $\mathbf{p} = (0, 0, 1)$. Consider the point $\mathbf{q} = (0, 1, 0)$ and the circle $C = \mathbf{Q} \cap F$, where

$$F := [(0, 0, 1)]^{\perp} = E(0, 0, 1, 0).$$

The lines

$$L_1 := L(\mathbf{q}, [(1, 0, 0)]) = \{\mathbf{q} + (1, 0, 0)t : t \in \mathbb{R}\},$$

$$L_2 := L(\mathbf{q}, [(1, r, 0)]) = \{\mathbf{q} + (1, r, 0)t : t \in \mathbb{R}\}$$

are in F and intersect $\mathbf{Q}$ at $\mathbf{q}$ only; see Figure 4.4.0.9.

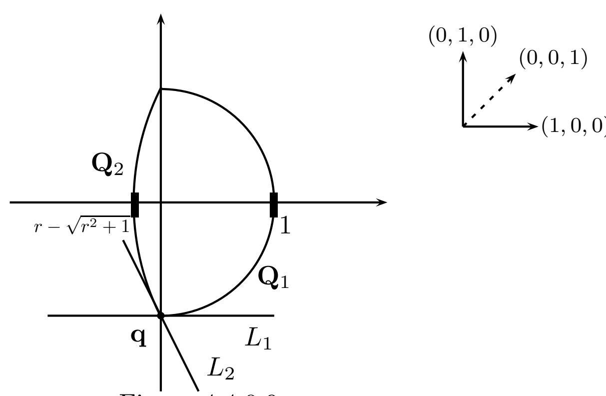


Figure 4.4.0.9

The planes

$$F_1 = \{\mathbf{q} + (\mathbf{p} - \mathbf{q})s + (1, 0, 0)t : s,\, t \in \mathbb{R}\},$$
$$F_2 = \{\mathbf{q} + (\mathbf{p} - \mathbf{q})s + (1, r, 0)t : s,\, t \in \mathbb{R}\}$$

intersect F along L_1 and L_2, respectively, and hence intersect C at $\mathbf{q}$ only. Furthermore, $\mathbf{q}$ lies on $\mathbf{Q} \cap F_1$ and $\mathbf{Q} \cap F_2$. See Figure 4.4.0.10.

Thus: we have a point $\mathbf{q}$ on a circle C, a point $\mathbf{p}$ not on C, and *two* different circles $\mathbf{Q} \cap F_1$, $\mathbf{Q} \cap F_2$ through $\mathbf{p}$ and intersecting C at $\mathbf{q}$ only. This means that $(\mathbf{Q}, \mathbf{G})$ is *not* a Möbius plane. □

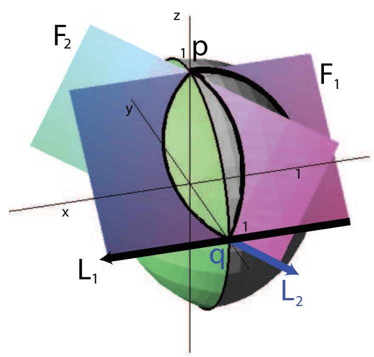


Figure 4.4.0.10: Two different tangent planes.

Chapter 5

The Flag Space

In the first part of this chapter we prove that the flag space of a (general, not necessarily embeddable) spherical circle plane is a 4-dimensional manifold. This result is stated but not proved[1] in the literature. In the second part, in analogy to our classification of the circle space of an embeddable spherical circle plane, we give a classification of its flag space, which is a new result.

5.1 Manifold Property of the Flag Space

The following is stated without proof in [6] (cf. [6], Proposition 3.2.17):

Theorem 5.1.1. *The flag space $\mathcal{F}$ of a spherical circle plane is a 4-dimensional manifold.*

To prove this, we use the following characterisation of spherical circle planes, as outlined in [6], Subsection 3.2.1.

[1] As far the author and his supervisor are aware.

5.1.1 Characterisation via Derived Planes

The *derived plane* of a spherical circle plane $(\mathcal{P},\mathcal{C})$ *at a point* $\mathbf{p} \in \mathcal{P}$ is the point-line geometry $(\mathcal{P}\backslash\mathbf{p},\mathcal{L})$, where $\mathcal{L}$ consists of all the circles in $\mathcal{C}$ passing through $\mathbf{p}$ but punctured at $\mathbf{p}$. Each line in $\mathcal{L}$ is a closed subset of $\mathcal{P}\backslash\mathbf{p}$ and is homeomorphic to $\mathbb{R}$; any two distinct points of $\mathcal{P}\backslash\mathbf{p}$ uniquely determine a line. Since $\mathcal{P} \approx \mathbb{S}^2_1$, we have that $\mathcal{P}\backslash\mathbf{p} \approx \mathbb{R}^2$.[2]

Of interest to us, however, is that the mapping from a spherical circle plane to its derived plane induces a homeomorphism from the circles in $\mathcal{C}$ not containing $\mathbf{p}$ to curves in $\mathbb{R}^2$ which are homeomorphic to $\mathbb{S}^1_1$ and each have the property that any line in $\mathbb{R}^2$ intersects them in at most two points. Let $\hat{\mathcal{C}}$ denote the collection of such curves obtained from $\mathcal{C}$ under the homeomorphism $\mathbf{P}\backslash\mathbf{p} \approx \mathbb{R}^2$; we henceforth refer to the curves in $\hat{\mathcal{C}}$ as *circles* of $\hat{\mathcal{C}}$. Denote $\hat{\mathcal{P}} := \mathbb{R}^2$. In this way, by deleting a distant point from $\mathcal{P}$, we may locally view a spherical circle plane in the $\mathbb{R}^2$ plane.

Recall that a flag is an ordered pair $(\mathbf{p}, C)$ where $\mathbf{p} \in \mathcal{P}$ and $C \in \mathcal{C}$ with $\mathbf{p} \in C$. Given a flag $(\mathbf{p}, C) \in \mathcal{F}$, a point $\mathbf{q} \neq \mathbf{p}$ and a circle $D \neq C$, since $\mathcal{P} \approx \mathbb{S}^2_1$ and $\mathcal{C}$ are Hausdorff[3], a neighbourhood of $(\mathbf{p}, C)$ in $(\mathcal{P}\backslash\mathbf{p} \times \mathcal{C}\backslash D) \cap \mathcal{F}$ is a neighbourhood of $(\mathbf{p}, C)$ in $\mathcal{F}$. This means we may use the derived plane characterisation to show that $\mathcal{F}$ is a 4-dimensional manifold. In the remainder of this section we consider points to lie in $\hat{\mathcal{P}}$ and circles to be those in $\hat{\mathcal{C}}$. Denote the flag space associated with $(\hat{\mathcal{P}},\hat{\mathcal{C}})$ by $\hat{\mathcal{F}}$.

Let $\hat{\alpha} : (\hat{\mathcal{P}}^3)_* \to \hat{\mathcal{C}}$ denote the joining map induced by α. Let $\hat{\Omega} : \hat{\mathcal{P}}^2 \to \widetilde{\hat{\mathcal{P}}^2}$ be the quotient map performing the identification

$$\hat{\Omega}(\mathbf{x}, \mathbf{y}) = \hat{\Omega}(\mathbf{u}, \mathbf{v}) \Leftrightarrow \{(\mathbf{x}, \mathbf{y}) = (\mathbf{u}, \mathbf{v}) \text{ or } (\mathbf{x}, \mathbf{y}) = (\mathbf{v}, \mathbf{u})\},$$

[2]In the literature, a point-line geometry with point and line sets having these properties is called an $\mathbb{R}^2$-*plane*.

[3]The circle space of a spherical circle plane is a 3-dimensional manifold; cf. [6], Proposition 3.2.14.

and let $\hat{\gamma} : (\hat{\mathcal{C}}^2)_* \to \widetilde{\hat{\mathcal{P}}^2}$ denote the intersection map induced by γ. These maps inherit the properties associated with a topological circle plane, namely, continuity of joining and stability of intersection.

We shall need the following.

Lemma 5.1.2 ([10], Theorem 2.5)**.** *Given a spherical circle plane $(\mathcal{P}, \mathcal{C})$, the connected components of the complements of circles that contain a point $\mathbf{p}$ form a neighbourhood basis of $\mathbf{p}$.*

Lemma 5.1.3 ([6], Theorem 3.2.2)**.** *Two distinct circles C, D of a spherical circle plane intersect in at most two points. If they intersect in two points $\mathbf{p}_1$ and $\mathbf{p}_2$, then they intersect transversally at these points. That is, for $i = 1, 2$, there are neighbourhoods U_i of $\mathbf{p}_i$ and homeomorphisms $f_i : U_i \to \mathbb{R}^2$ such that $f_i(C \cap U_i) = \mathbb{R} \times 0$ and $f_i(D \cap U_i) = 0 \times \mathbb{R}$.*

We devote the remainder of this section to proving Theorem 5.1.1.

Let $(\mathbf{p}, C) \in \hat{\mathcal{F}}$ and choose $r > 0$ so that C is not contained in $\mathcal{B}(\mathbf{p}, r)$. Applying Lemma 5.1.2 to the neighbourhood $\mathcal{B}(\mathbf{p}, r)$ of $\mathbf{p}$, there is a circle $D \in \hat{\mathcal{C}}$ and a connected component B of D^c such that $\mathbf{p} \in B \subseteq \mathcal{B}(\mathbf{p}, r)$. Note that, since $\mathcal{B}(\mathbf{p}, r)$ is bounded, by the Jordan-Brouwer separation theorem (Theorem A.1.13) B is necessarily the bounded component of D^c; see Figure 5.1.1.1. Furthermore, D intersects C at precisely two points $\mathbf{p}_1$ and $\mathbf{p}_2$. Therefore, in light of Lemma 5.1.3, there are neighbourhoods U_1, U_2 of $\mathbf{p}_1$ and $\mathbf{p}_2$ in $\hat{\mathcal{P}}$, respectively, such that $I_1 := U_1 \cap D$ and $I_2 := U_2 \cap D$ are disjoint and each homeomorphic to $\mathbb{R}$.

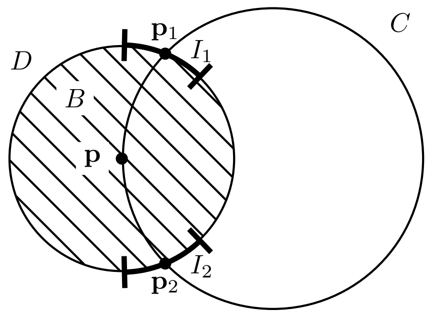


Figure 5.1.1.1

Lemma 5.1.4. *The quotient map* $\hat{\Omega} : \hat{\mathcal{P}}^2 \to \widetilde{\hat{\mathcal{P}}^2}$ *is an open map.*

Proof. It suffices to show that if X, Y are open subsets of $\hat{\mathcal{P}}$, then $\hat{\Omega}(X \times Y)$ is open in $\widetilde{\hat{\mathcal{P}}^2}$. By the definition of $\hat{\Omega}$, we have $\hat{\Omega}(X \times Y) = \hat{\Omega}(Y \times X)$, so

$$\hat{\Omega}(X \times Y) = \hat{\Omega}\big((X \times Y) \cup (Y \times X)\big).$$

And clearly

$$\hat{\Omega}^{-1}\Big(\hat{\Omega}\big((X \times Y) \cup (Y \times X)\big)\Big) = (X \times Y) \cup (Y \times X).$$

Hence

$$\hat{\Omega}^{-1}\big(\hat{\Omega}(X \times Y)\big) = (X \times Y) \cup (Y \times X)$$

is open in $\hat{\mathcal{P}}^2$, so $\hat{\Omega}(X \times Y)$ is open in $\widetilde{\hat{\mathcal{P}}^2}$. $\square$

Lemma 5.1.5. *The restriction-corestriction*

$$\hat{\omega} := \hat{\Omega}|_{I_1 \times I_2} : I_1 \times I_2 \to \hat{\Omega}(I_1 \times I_2) =: \widetilde{I_1 \times I_2}$$

of $\hat{\Omega}$ *is a homeomorphism.*

Proof. Since $\hat{\omega}$ is a restriction-corestriction of the continuous map $\hat{\Omega}$, it is continuous and surjective. Let us now verify that the inverse map

$$\hat{\omega}^{-1} : \hat{\omega}(\mathbf{x}, \mathbf{y}) = \{\mathbf{x}, \mathbf{y}\} \mapsto (\mathbf{x}, \mathbf{y})$$

is well-defined. Let $(\mathbf{x}, \mathbf{y})$, $(\mathbf{u}, \mathbf{v}) \in I_1 \times I_2$ be such that $\hat{\omega}(\mathbf{x}, \mathbf{y}) = \hat{\omega}(\mathbf{u}, \mathbf{v})$. Then, as I_1 and I_2 are disjoint, we have

$$(\mathbf{x}, \mathbf{y}) \in \{(\mathbf{u}, \mathbf{v}), (\mathbf{v}, \mathbf{u})\} \cap (I_1 \times I_2) = (\mathbf{u}, \mathbf{v});$$

hence $\hat{\omega}^{-1}$ is well-defined.

Lastly, we verify that $\hat{\omega}^{-1}$ is continuous. Suppose that $(\{\mathbf{x}_n, \mathbf{y}_n\})$ is a sequence in $\widetilde{I_1 \times I_2}$ converging to an element $\{\mathbf{x}, \mathbf{y}\}$, where $\mathbf{x} \in I_1$, $\mathbf{y} \in I_2$. For each n, $(\mathbf{x}_n, \mathbf{y}_n)$ is exactly one of

$$(\{\mathbf{x}_n, \mathbf{y}_n\} \cap I_1, \{\mathbf{x}_n, \mathbf{y}_n\} \cap I_2) \quad \text{or} \quad (\{\mathbf{x}_n, \mathbf{y}_n\} \cap I_2, \{\mathbf{x}_n, \mathbf{y}_n\} \cap I_1),$$

so we may define a sequence $(\{\mathbf{x}_n', \mathbf{y}_n'\})$ in $\widetilde{I_1 \times I_2}$ by

$$\mathbf{x}_n' = \{\mathbf{x}_n, \mathbf{y}_n\} \cap I_1, \quad \mathbf{y}_n' = \{\mathbf{x}_n, \mathbf{y}_n\} \cap I_2.$$

Then $\{\mathbf{x}_n', \mathbf{y}_n'\} = \{\mathbf{x}_n, \mathbf{y}_n\}$ for each n and, moreover, $(\mathbf{x}_n', \mathbf{y}_n') \to (\mathbf{x}, \mathbf{y})$ as $n \to \infty$. Thus

$$\hat{\omega}^{-1}(\{\mathbf{x}_n, \mathbf{y}_n\}) = \hat{\omega}^{-1}(\{\mathbf{x}_n', \mathbf{y}_n'\}) = (\mathbf{x}_n', \mathbf{y}_n') \to (\mathbf{x}, \mathbf{y}) = \hat{\omega}^{-1}(\{\mathbf{x}, \mathbf{y}\}),$$

and we deduce that $\hat{\omega}^{-1}$ is continuous. Hence $\hat{\omega}$ is a homeomorphism. □

We continue to use the notation depicted in Figure 5.1.1.1. Choose $r' > 0$ so that $U := \mathcal{B}(\mathbf{p}, r')$ lies in B (which is a neighbourhood of $\mathbf{p}$ in $\hat{\mathcal{P}}$); see Figure

5.1.1.2. Recall that the intersection map

$$\hat{\gamma} : (\hat{\mathcal{C}}^2)_* \to \widetilde{\hat{\mathcal{P}}^2} : (C_1, C_2) \mapsto C_1 \cap C_2$$

is continuous. Define the map

$$\hat{\gamma}_D : \{C' \in \hat{\mathcal{C}} : |C' \cap D| = 2\} \to \hat{\Omega}(D^2)$$

by

$$\hat{\gamma}_D(C') = D \cap C'.$$

Note that $\hat{\gamma}_D$ is the composition of $\hat{\gamma}$ with the inclusion map $C' \mapsto (C', D)$, so is continuous. Let

$$V := \hat{\gamma}_D^{-1}(\widetilde{I_1 \times I_2}) \subseteq \hat{\mathcal{C}}.$$

We proceed to show that $(U \times V) \cap \hat{\mathcal{F}}$ is a neighbourhood of $(\mathbf{p}, C)$ in $\hat{\mathcal{F}}$ and is homeomorphic to $\mathbb{R}^4$.

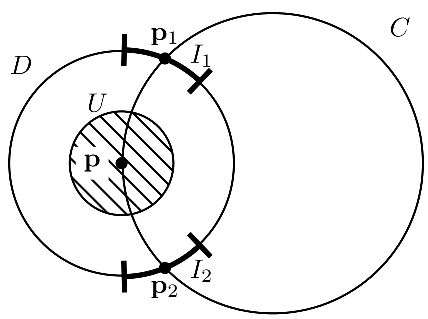


Figure 5.1.1.2

Proposition 5.1.6. *The subset $(U \times V) \cap \hat{\mathcal{F}}$ is a neighbourhood of $(\mathbf{p}, C)$ in $\hat{\mathcal{F}}$.*

Proof. Since $\hat{\gamma}_D(C) = D \cap C \in \widetilde{I_1 \times I_2}$, we have $C \in \hat{\gamma}_D^{-1}(\widetilde{I_1 \times I_2}) = V$. So, as $\mathbf{p} \in \mathcal{B}(\mathbf{p}, r') = U$, we have $(\mathbf{p}, C) \in U \times V$.

We now verify that $U \times V$ is open in $\hat{\mathcal{P}} \times \hat{\mathcal{C}}$. Clearly U is open in $\hat{\mathcal{P}}$.

In order to show that V is open in $\hat{\mathcal{C}}$, we first show that

$$\widetilde{I_1 \times I_2} = \hat{\Omega}(I_1 \times I_2) = \hat{\Omega}(U_1 \times U_2) \cap \hat{\Omega}(D^2).$$

On the one hand, $\hat{\Omega}(I_1 \times I_2) \subseteq \hat{\Omega}(U_1 \times U_2) \cap \hat{\Omega}(D^2)$ because $I_1 \times I_2 = (U_1 \times U_2) \cap D^2$.

Conversely, let $\{\mathbf{x}, \mathbf{y}\} \in \hat{\Omega}(U_1 \times U_2) \cap \hat{\Omega}(D^2)$. Then $(\mathbf{x}, \mathbf{y}) \in \{(\mathbf{u}, \mathbf{v}), (\mathbf{v}, \mathbf{u})\}$ for some $\mathbf{u} \in U_1$, $\mathbf{v} \in U_2$ such that $\{\mathbf{u}, \mathbf{v}\} \in \hat{\Omega}(D^2)$. That is, $(\mathbf{x}, \mathbf{y}) \in \{(\mathbf{u}, \mathbf{v}), (\mathbf{v}, \mathbf{u})\}$ for some $\mathbf{u} \in U_1 \cap D = I_1$, $\mathbf{v} \in U_2 \cap D = I_2$. Hence $\hat{\Omega}(\mathbf{x}, \mathbf{y}) \in \hat{\Omega}(I_1 \times I_2)$ and we deduce that $\hat{\Omega}(U_1 \times U_2) \cap \hat{\Omega}(D^2) \subseteq \hat{\Omega}(I_1 \times I_2)$.

Now, since $\hat{\Omega}$ is an open map (Lemma 5.1.4), $\hat{\Omega}(U_1 \times U_2)$ is open in $\hat{\Omega}(\hat{\mathcal{P}}^2)$. Thus

$$\widetilde{I_1 \times I_2} = \hat{\Omega}(U_1 \times U_2) \cap \hat{\Omega}(D^2)$$

is open in the subspace $\hat{\Omega}(D^2)$. By the continuity of $\hat{\gamma}_D$, then, we obtain that $V = \hat{\gamma}_D^{-1}(\widetilde{I_1 \times I_2})$ is open in $\{C' \in \hat{\mathcal{C}} : |C' \cap D| = 2\}$.

Now, if $C' \in \hat{\mathcal{C}}$ such that $|C' \cap D| = 2$, by the stability of intersection there are open neighbourhoods W_1 of C' and W_2 of D such that any two circles from these neighbourhoods intersect in 2 points. In particular, since $D \in W_2$, all circles in W_1 intersect D in 2 points. This shows that $\{C' \in \hat{\mathcal{C}} : |C' \cap D| = 2\}$ is open in $\hat{\mathcal{C}}$ and we deduce that V is open in $\hat{\mathcal{C}}$. This completes our proof that $U \times V$ is open in $\hat{\mathcal{P}} \times \hat{\mathcal{C}}$. $\square$

Lemma 5.1.7. *The map*

$$\Lambda : U \times I_1 \times I_2 \to (U \times V) \cap \hat{\mathcal{F}}$$
$$: (\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto (\mathbf{x}, \hat{\alpha}(\mathbf{x}, \mathbf{y}, \mathbf{z}))$$

is a homeomorphism, with inverse

$$\begin{aligned}\Lambda^{-1} &: (U \times V) \cap \hat{\mathcal{F}} \to U \times I_1 \times I_2 \\ &: (\mathbf{x}, E) \mapsto \big(\mathbf{x}, \hat{\omega}^{-1}(\hat{\gamma}_D(E))\big).\end{aligned}$$

Proof. Let us first verify that Λ is well-defined. Let $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in U \times I_1 \times I_2$. Since U, I_1 and I_2 are mutually disjoint, $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are mutually distinct and so $\hat{\alpha}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a circle; hence $(\mathbf{x}, \hat{\alpha}(\mathbf{x}, \mathbf{y}, \mathbf{z})) \in \hat{\mathcal{F}}$. Furthermore, since

$$\{\mathbf{y}, \mathbf{z}\} \subseteq \hat{\alpha}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \cap D,$$

we have

$$\hat{\gamma}_D(\hat{\alpha}(\mathbf{x}, \mathbf{y}, \mathbf{z})) = \hat{\gamma}(D, \hat{\alpha}(\mathbf{x}, \mathbf{y}, \mathbf{z})) = \{\mathbf{y}, \mathbf{z}\} \in \widetilde{I_1 \times I_2},$$

and thus

$$\hat{\alpha}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \hat{\gamma}_D^{-1}(\widetilde{I_1 \times I_2}) = V.$$

This shows that $(\mathbf{x}, \hat{\alpha}(\mathbf{x}, \mathbf{y}, \mathbf{z})) \in (U \times V) \cap \hat{\mathcal{F}}$, so Λ is well-defined.

Since $\Lambda = p_1 \times \hat{\alpha}$, where $p_1 : (\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto \mathbf{x}$ is the continuous projection map, we see that Λ is continuous.

That Λ^{-1} maps into $U \times I_1 \times I_2$ is immediate from the definitions of $\hat{\gamma}_D$ and V. Since

$$\Lambda^{-1} = p_1' \times (\hat{\omega}^{-1} \circ \hat{\gamma}_D),$$

where

$$p_1' : (\mathbf{x}, C) \mapsto \mathbf{x}$$

is the continuous projection map, we see that Λ^{-1} is continuous.

Finally, we verify that Λ and Λ^{-1} are indeed inverses. We have

$$\begin{aligned}\Lambda^{-1}\circ\Lambda : (\mathbf{x},\mathbf{y},\mathbf{z}) &\overset{\Lambda}{\longmapsto} (\mathbf{x},\hat{\alpha}(\mathbf{x},\mathbf{y},\mathbf{z}))\\ &\xmapsto{\Lambda^{-1}} (\mathbf{x},\hat{\omega}^{-1}(\hat{\gamma}(D,\hat{\alpha}(\mathbf{x},\mathbf{y},\mathbf{z}))))\\ &= (\mathbf{x},\hat{\omega}^{-1}(\{\mathbf{y},\mathbf{z}\})) = (\mathbf{x},\mathbf{y},\mathbf{z}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\Lambda\circ\Lambda^{-1} : (\mathbf{x},E) \xmapsto{\Lambda^{-1}} (\mathbf{x},\hat{\omega}^{-1}(\hat{\gamma}(D,E))) = (\mathbf{x},\mathbf{y},\mathbf{z}) &\overset{\Lambda}{\longmapsto} (\mathbf{x},\hat{\alpha}(\mathbf{x},\mathbf{y},\mathbf{z}))\\ &= (\mathbf{x},E)\end{aligned}$$

since $\mathbf{x},\mathbf{y},\mathbf{z}\in E$. Whence Λ^{-1} is the inverse mapping of Λ and this shows that Λ is a homeomorphism. □

Proposition 5.1.8. *The subset* $(U\times V)\cap\hat{\mathcal{F}}$ *is homeomorphic to* $\mathbb{R}^4$*.*

Proof. Since $U=\mathcal{B}(\mathbf{p},r')\approx\mathbb{R}^2$, and I_1 and I_2 are each homeomorphic to $\mathbb{R}$, we have that $U\times I_1\times I_2\approx\mathbb{R}^4$. Hence, by Lemma 5.1.7, the subset $(U\times V)\cap\hat{\mathcal{F}}$ of $\hat{\mathcal{F}}$ is homeomorphic to $\mathbb{R}^4$. □

We may now complete the proof of Theorem 5.1.1.

By Propositions 5.1.6 and 5.1.8, each element $(\mathbf{p},C)\in\hat{\mathcal{F}}$ has a neighbourhood of the form $U\times I_1\times I_2$, which is homeomorphic to $\mathbb{R}^4$.

It remains for us to show that $\mathcal{F}$ is Hausdorff and second countable. Since $\mathcal{P}$ and $\mathcal{C}$ are Hausdorff spaces, so is $\mathcal{F}\subseteq\mathcal{P}\times\mathcal{C}$.

Now, since $\mathcal{P}\approx\mathbb{S}^2_1$ is homeomorphic to a subspace of $\mathbb{R}^3$, we have that $(\mathcal{P}^3)_*\subseteq\mathcal{P}^3$ is homeomorphic to a subspace of the second-countable space $(\mathbb{R}^3)^3$, so is itself second countable.

By [10], Corollary 2.8, the joining map α is open (in addition to being continuous). Hence, as the projection map $q_1:(\mathbf{x},\mathbf{y},\mathbf{z})\mapsto\mathbf{x}$ is continuous and open, we have that the map

$$q_1 \times \alpha : (\mathbf{x}, \mathbf{y}, \mathbf{z}) \mapsto (\mathbf{x}, \alpha(\mathbf{x}, \mathbf{y}, \mathbf{z}))$$

is continuous and open. Thus

$$\mathcal{F} = (q_1 \times \alpha)((\mathcal{P}^3)_*)$$

is the continuous, open image of a second-countable space, so is second-countable (cf. Theorem A.1.12).

This completes the proof that $\mathcal{F}$ is a 4-dimensional manifold.

5.2 The Flag Space of an ESCP

Let $\mathbf{F}$, $\mathcal{F}_c$ denote the flag spaces of an embeddable spherical circle plane $(\mathbf{P}, \mathbf{C})$ and the classical flat Möbius plane $(\mathbb{S}_1^2, \mathcal{C}_c)$, respectively. In this section we first show that $\mathbf{F}$ and $\mathcal{F}_c$ are homeomorphic. We then give a description of the topological structure of $\mathcal{F}_c$.

5.2.1 A Homeomorphism between Flag Spaces

In this subsection we prove the following.

Theorem 5.2.1. *The flag space* $\mathbf{F}$ *of an embeddable spherical circle plane* $(\mathbf{P}, \mathbf{C})$ *is homeomorphic to the flag space* $\mathcal{F}_c$ *of the classical flat Möbius plane* $(\mathbb{S}_1^2, \mathcal{C}_c)$.

We begin by describing the desired map between $\mathbf{F}$ and $\mathcal{F}_c$.

Recall that we denote the bounded component of $\mathbb{R}^3 \backslash \mathbf{P}$ by $\mathbf{B}$. Let us restate the definition of the map η and introduce the map η_c:

$$\eta_c : \mathcal{B}(\mathbf{0},1) \times \mathbb{S}^2_1 \to \mathbb{S}^2_1 : (\mathbf{x},\mathbf{u}) \mapsto \mathcal{R}(\mathbf{x},\mathbf{u}) \cap \mathbb{S}^2_1;$$

$$\eta : \mathbf{B} \times \mathbb{S}^2_1 \to \mathbf{P} : (\mathbf{x},\mathbf{u}) \mapsto \mathcal{R}(\mathbf{x},\mathbf{u}) \cap \mathbf{P};$$

these are continuous by Proposition 3.4.14.

Note that the distance map $\mathfrak{f}_{\mathbb{S}^2_1}$ for $\mathbb{S}^2_1$ has the constant value 1, and the maximal map $\mathfrak{h}_{\mathbb{S}^2_1}$ is the identity on $\mathbb{S}^2_1$. Abbreviate the corresponding maps for $\mathbf{P}$ by $\mathfrak{f} := \mathfrak{f}_{\mathbf{P}}$ and $\mathfrak{h} := \mathfrak{h}_{\mathbf{P}}$.

Consider first a flag in $\mathbf{F}$ of the form $(\mathbf{p}, E(\mathbf{u}{\times}d))$, where $0 < d < \mathfrak{f}(\mathbf{u})$; let $d' = \frac{d}{\mathfrak{f}(\mathbf{u})} < 1$. The point $d'\mathfrak{h}(\mathbf{u})$ lies on the open line segment between the origin $\mathbf{0}$ and the point $\mathfrak{h}(\mathbf{u}) \in \mathbf{P}$, so lies in $\mathbf{B}$. Hence $\mathbf{p}$ is precisely the point of intersection with $\mathbf{P}$ of the open ray $\mathcal{R}(d'\mathfrak{h}(\mathbf{u}), \mathbf{w})$ emanating from $d'\mathfrak{h}(\mathbf{u})$ in the direction $\mathbf{w} := \frac{\mathbf{p}-d'\mathfrak{h}(\mathbf{u})}{||\mathbf{p}-d'\mathfrak{h}(\mathbf{u})||}$. By definition, $\mathfrak{h}(\mathbf{u}) \cdot \mathbf{u} = \mathfrak{f}(\mathbf{u})$, so

$$d'\mathfrak{h}(\mathbf{u}) \cdot \mathbf{u} = \tfrac{d}{\mathfrak{f}(\mathbf{u})}\mathfrak{f}(\mathbf{u}) = d.$$

That is, $d'\mathfrak{h}(\mathbf{u}) \in E(\mathbf{u}{\times}d)$, so $\mathcal{R}(d'\mathfrak{h}(\mathbf{u}), \mathbf{w})$ lies in the plane $E(\mathbf{u}{\times}d)$. See Figure 5.2.1.1.

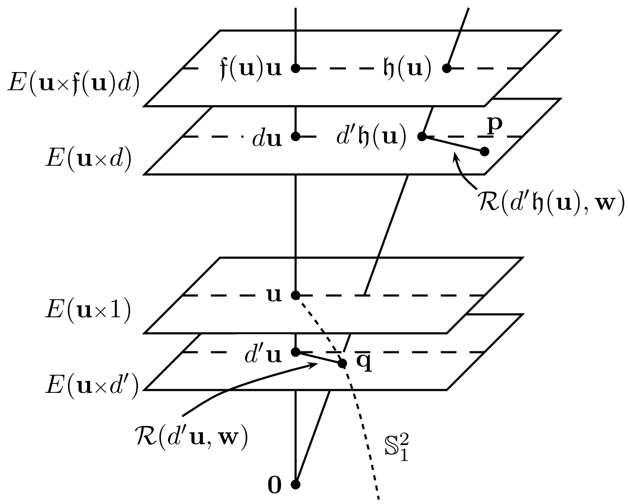


Figure 5.2.1.1

Now, the point $d'\mathbf{u}$ lies in $\mathcal{B}(\mathbf{0},1)$ and on the plane $E(\mathbf{u}\times d')$. We map $\mathbf{p}$ to the point of intersection of the ray $\mathcal{R}(d'\mathbf{u},\mathbf{w})$ with $\mathbb{S}^2_1$. The flag $(\mathbf{p},E(\mathbf{u}\times d))$ in $\mathbf{F}$ is then defined to map to the flag $(\mathbf{q},E(\mathbf{u}\times d'))$ in $\mathcal{F}_c$. Notice this mechanism is bijective: given a flag $(\mathbf{q},E(\mathbf{u}\times d'))$ in $\mathcal{F}_c$, we map it to $(\mathbf{p},E(\mathbf{u}\times d'\mathfrak{f}(\mathbf{u}))$, where

$$\mathbf{p}=\eta\left(d'\mathfrak{h}(\mathbf{u}),\mathbf{w}\right),$$

with $\mathbf{w}:=\frac{\mathbf{q}-d'\mathbf{u}}{||\mathbf{q}-d'\mathbf{u}||}$.

On the other hand, now consider a flag in $\mathbf{F}$ of the form $(\mathbf{p},E(\mathbf{u}\times 0))$. Since $\mathbf{0}\in\mathbf{B}$, we may bijectively map $\mathbf{p}$ to $\frac{\mathbf{p}}{||\mathbf{p}||}$ using that $\frac{\mathbf{p}}{||\mathbf{p}||}=\eta_c(\mathbf{0},\frac{\mathbf{p}}{||\mathbf{p}||})$ and $\mathbf{p}=\eta(\mathbf{0},\frac{\mathbf{p}}{||\mathbf{p}||})$. Thence $(\mathbf{p},E(\mathbf{u}\times 0))\in\mathbf{F}$ is defined to map to $(\frac{\mathbf{p}}{||\mathbf{p}||},E(\mathbf{u}\times 0))\in\mathcal{F}_c$; conversely, a flag $(\mathbf{q},E(\mathbf{u}\times 0))\in\mathcal{F}_c$ is mapped to $(\mathbf{p},E(\mathbf{u}\times 0))$, where $\mathbf{p}=\eta(\mathbf{0},\mathbf{q})$.

We proceed to formalise this mapping and show that it is a homeomorphism. Define $\tau:\mathbb{R}^3\backslash\mathbf{0}\to\mathbb{R}^3\backslash\mathbf{0}$ by $\tau(\mathbf{x})=\mathfrak{f}(\mathbf{u})\mathbf{x}$, where $\mathbf{u}=\frac{\mathbf{x}}{||\mathbf{x}||}$. Then τ is continuous and has continuous inverse $\mathbf{x} \mapsto \frac{\mathbf{x}}{\mathfrak{f}(\mathbf{u})}$, and

$$\frac{\tau(\mathbf{x})}{||\tau(\mathbf{x})||} = \frac{\mathbf{x}}{||\mathbf{x}||} = \frac{\tau^{-1}(\mathbf{x})}{||\tau^{-1}(\mathbf{x})||}$$

for each $\mathbf{x} \in \mathbb{R}^3 \backslash \mathbf{0}$. Let $(\mathbf{x}_n)$ be sequence in $\mathbb{R}^3 \backslash \mathbf{0}$ such that $\frac{1}{||\mathbf{x}_n||} \to 0$; put $\mathbf{u}_n = \frac{\mathbf{x}_n}{||\mathbf{x}_n||}$ for each n. By the continuity of $\mathfrak{f}$ and the compactness of $\mathbb{S}^1_1$, we have that $(\mathfrak{f}(\mathbf{u}_n))$ is bounded away from 0. Thus:

$$\frac{1}{||\tau(\mathbf{x}_n)||} = \frac{1}{\mathfrak{f}(\mathbf{u}_n)||\mathbf{x}_n||} \to 0 \quad \text{and} \quad \frac{1}{||\tau^{-1}(\mathbf{x}_n)||} = \frac{\mathfrak{f}(\mathbf{u}_n)}{||\mathbf{x}_n||} \to 0$$

as $n \to \infty$.

Recall that

$$\mathbf{C} = \left\{[\mathbf{x} \times 0] : \mathbf{x} \in \mathbb{R}^3 \backslash \mathbf{0}\right\} \cup \left\{[\mathbf{x} \times 1] : \mathbf{x} \in \mathbb{R}^3 \backslash \mathbf{0}, \tfrac{1}{||\mathbf{x}||} < \mathfrak{f}\left(\tfrac{\mathbf{x}}{||\mathbf{x}||}\right)\right\}$$

and

$$\mathcal{C}_c = \left\{[\mathbf{x} \times 0] : \mathbf{x} \in \mathbb{R}^3 \backslash \mathbf{0}\right\} \cup \left\{[\mathbf{x} \times 1] : \mathbf{x} \in \mathbb{R}^3 \backslash \mathbf{0}, \tfrac{1}{||\mathbf{x}||} < 1\right\}.$$

Let $T : \mathbf{C} \to \mathcal{C}_c$ be the map defined on $\mathcal{Y}_1 \cap \mathbf{C}$ and $\mathcal{Y}_2 \cap \mathbf{C}$ by

$$\begin{aligned} &[\mathbf{x} \times 0] \mapsto [\mathbf{x} \times 0], \\ &[\mathbf{x} \times 1] \mapsto [\tau(\mathbf{x}) \times 1] = [\mathfrak{f}(\mathbf{u})\mathbf{x} \times 1], \end{aligned}$$

respectively. Note that T is well-defined since, putting $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$, we have that $\frac{1}{||\mathbf{x}||} < \mathfrak{f}(\mathbf{u})$ if and only if

$$\frac{1}{||\mathfrak{f}(\mathbf{u})\mathbf{x}||} = \frac{1}{\mathfrak{f}(\mathbf{u})} \frac{1}{||\mathbf{x}||} < 1.$$

Since T is a restriction-corestriction of a homeomorphism of the form in Theorem 4.3.5, it is a homeomorphism.

Lemma 5.2.2. *The map* $G : \mathbf{F} \to \mathcal{B}(\mathbf{0}, 1) \times \mathbb{S}_1^2$*, given by*

$$
\begin{aligned}
(\mathbf{p}, [\mathbf{x} \times 0]) &\mapsto (\mathbf{0}, \tfrac{\mathbf{p}}{||\mathbf{p}||}),\\
(\mathbf{p}, [\mathbf{x} \times 1]) &\mapsto \left(\tfrac{d}{\mathfrak{f}(\mathbf{u})}\mathbf{u}, \frac{\mathbf{p} - \frac{d}{\mathfrak{f}(\mathbf{u})}\mathfrak{h}(\mathbf{u})}{\left\|\mathbf{p} - \frac{d}{\mathfrak{f}(\mathbf{u})}\mathfrak{h}(\mathbf{u})\right\|} \right),
\end{aligned}
$$

where $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$ *and* $d = \frac{1}{||\mathbf{x}||} < \mathfrak{f}(\mathbf{u})$*, is continuous.*

Proof. Let $(\mathbf{p}_n)$ be a sequence in $\mathbf{P}$ converging to $\mathbf{p} \in \mathbf{P}$ and let (Y_n) be a sequence in $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$ converging to Y. Recall that we denote

$$
\begin{aligned}
\mathcal{Y}_1 &= \left\{[\mathbf{x} \times 0] : \mathbf{x} \in \mathbb{R}^3\backslash\mathbf{0}\right\} \text{ and}\\
\mathcal{Y}_2 &= \left\{[\mathbf{x} \times 1] : \mathbf{x} \in \mathbb{R}^3\backslash\mathbf{0}\right\}.
\end{aligned}
$$

Let (Y_n) be a sequence in $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$ converging to Y. If $Y \in \mathcal{Y}_2$, since $\mathcal{Y}_2$ is open we have that $Y_n \in \mathcal{Y}_2$ for sufficiently large n. Since the mapping

$$
[\mathbf{x} \times 1] \overset{\xi}{\mapsto} \mathbf{u} \times d \mapsto \frac{d}{\mathfrak{f}(\mathbf{u})\mathfrak{h}(\mathbf{u})},
$$

where $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$ and $d = \frac{1}{||\mathbf{x}||}$, is the composition of continuous maps, it follows that G is continuous on $\mathcal{Y}_2$. Whence $G(Y_n) \to G(Y)$.

Now suppose that $Y \in \mathcal{Y}_1$; put $Y = [\mathbf{x} \times 0]$ and $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$.

If $Y_n \in \mathcal{Y}_1$ for only finitely many n, then by removing these elements from the sequence and relabelling we have $Y_n \in \mathcal{Y}_2$ for all n. Write $Y_n = [\mathbf{x}_n \times 1]$ and let $(\mathbf{u}_n)$ and (d_n) be the sequences in $\mathbb{S}_1^2$, $\mathbb{R}_{>0}$ given by $\mathbf{u}_n = \frac{\mathbf{x}_n}{||\mathbf{x}_n||}$ and $d_n = \frac{1}{||\mathbf{x}_n||}$ for each n, respectively. Then

$$
[\mathbf{u}_n \times d_n] = [\mathbf{x}_n \times 1] \to [\mathbf{x} \times 0]
$$

and so by Lemma 4.3.2 we have that $\mathbf{u}_n \to \mathbf{u}$ and $d_n \to 0$. Since $\mathfrak{f}$ is continuous and $(\mathfrak{f}(\mathbf{u}_n))$ is bounded above and below away from 0, we have that $\frac{d_n}{f(\mathbf{u}_n)} \to 0$. Combining this with the continuity of $\mathfrak{h}$, we obtain that

$$\begin{aligned} G(\mathbf{p}_n, [\mathbf{x}_n \times 1]) &= \left(\tfrac{d_n}{\mathfrak{f}(\mathbf{u}_n)} \mathbf{u}_n, \frac{\mathbf{p}_n - \frac{d_n}{\mathfrak{f}(\mathbf{u}_n)} \mathfrak{h}(\mathbf{u}_n)}{\left\| \mathbf{p}_n - \frac{d_n}{\mathfrak{f}(\mathbf{u}_n)} \mathfrak{h}(\mathbf{u}_n) \right\|} \right) \\ &\to \left(\mathbf{0}, \tfrac{\mathbf{p}-\mathbf{0}}{||\mathbf{p}-\mathbf{0}||} \right) = G([\mathbf{x} \times 0]). \end{aligned}$$

Suppose now that $Y_n \in \mathcal{Y}_1$ for infinitely many n. If there is an $N > 0$ such that $Y_n \in \mathcal{Y}_1$ for all $n > N$, putting $Y_n = [\mathbf{x}_n \times 0]$, we have

$$G(\mathbf{p}_n, [\mathbf{x}_n \times 0]) = G\left(\mathbf{0}, \tfrac{\mathbf{p}_n}{||\mathbf{p}_n||}\right) \to \left(\mathbf{0}, \tfrac{\mathbf{p}}{||\mathbf{p}||}\right) = G([\mathbf{x} \times 0]).$$

On the other hand, let $(Y_m)_{m \in N_1}$ and $(Y_{m'})_{m' \in N_2}$ be infinite subsequences of (Y_n) corresponding to the elements in $\mathcal{Y}_1$ and $\mathcal{Y}_2$, respectively, where N_1 and N_2 partition $\mathbb{N}$. By the cases "$Y \in \mathcal{Y}_1$ and $Y_n \in \mathcal{Y}_1$ for all n" and "$Y \in \mathcal{Y}_1$ and $Y_n \in \mathcal{Y}_2$ for all n" just considered, respectively, it is apparent that $G(Y_m) \xrightarrow{m \to \infty} G(Y)$ and $G(Y_{m'}) \xrightarrow{m' \to \infty} G(Y)$. Whence, letting $\varepsilon > 0$, there are $K, L > 0$ such that $\varrho_3(G(Y_m), G(Y)) < \varepsilon$ and $\varrho_3(G(Y_{m'}), G(Y)) < \varepsilon$ for all $m > K$, $m' > L$. For each $n > \max\{K, L\}$, then, we have

$$\begin{aligned} \varrho_3(G(Y_n), G(Y)) &= \begin{cases} \varrho_3(G(Y_m), G(Y)) & \text{if } n \in \{m \in N_1 : m > K\}, \text{ or} \\ \varrho_3(G(Y_{m'}), G(Y)) & \text{if } n \in \{m' \in N_2 : m' > L\} \end{cases} \\ &< \varepsilon, \end{aligned}$$

so $G(Y_n) \to G(Y)$. This shows that G is continuous. □

Define the map $G^c : \mathcal{F}_c \to \mathbf{B} \times \mathbb{S}^2_1$ by

$$\begin{aligned}
(\mathbf{p}, [\mathbf{x} \times 0]) &\mapsto (\mathbf{0}, \mathbf{p}), \\
(\mathbf{p}, [\mathbf{x} \times 1]) &\mapsto \left(d\mathfrak{h}(\mathbf{u}), \tfrac{\mathbf{p} - d\mathbf{u}}{\|\mathbf{p} - d\mathbf{u}\|}\right),
\end{aligned}$$

where $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$ and $d = \frac{1}{||\mathbf{x}||} < 1$.

Lemma 5.2.3. *G^c is continuous.*

Proof. The proof is completely analogous to that of Lemma 5.2.2. □

We are now in a position to formally define the map $F : \mathbf{F} \to \mathcal{F}_c$ described at the beginning of this section. Let

$$\begin{aligned}
F_1 &:= \eta_c \circ G : \mathbf{F} \to \mathbb{S}^2_1 \\
&: \begin{cases} (\mathbf{p}, [\mathbf{x} \times 0]) & \mapsto \eta_c(\mathbf{0}, \mathbf{p}), \\ (\mathbf{p}, [\mathbf{x} \times 1]) & \mapsto \eta_c\left(d'\mathbf{u}, \frac{\mathbf{p} - d'\mathfrak{h}(\mathbf{u})}{\left\|\mathbf{p} - d'\mathfrak{h}(\mathbf{u})\right\|}\right), \end{cases}
\end{aligned}$$

and let

$$\begin{aligned}
F_2 &:= T \circ \mathrm{Proj}_2 : \mathbf{F} \to \mathcal{C}_c \\
&: \begin{cases} (\mathbf{p}, [\mathbf{x} \times 0]) & \mapsto [\mathbf{x} \times 0], \\ (\mathbf{p}, [\mathbf{x} \times 1]) & \mapsto [\mathfrak{f}(\mathbf{u})\mathbf{x} \times 1], \end{cases}
\end{aligned}$$

where $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$, $d = \frac{1}{||\mathbf{x}||}$, $d' = \frac{d}{\mathfrak{f}(\mathbf{u})} < 1$ and $\mathrm{Proj}_2 : \mathcal{F} \to \mathbf{C}$ is the projection map

$$(\mathbf{p}, [\mathbf{x} \times d]) \mapsto [\mathbf{x} \times d].$$

We define $F := F_1 \times F_2$. Since η_c, G, T and Proj_2 are each continuous, so is F. Noting that $[\mathbf{x} \times 1] = [\mathbf{u} \times d]$ and $[\mathfrak{f}(\mathbf{u})\mathbf{x} \times 1] = [\mathbf{u} \times d']$, we may describe

$F : \mathbf{F} \to \mathcal{F}_c$ explicitly by

$$\begin{cases} (\mathbf{p}, [\mathbf{u} \times 0]) & \mapsto (\eta_c(\mathbf{0}, \mathbf{p}), [\mathbf{u} \times 0])\,, \\ (\mathbf{p}, [\mathbf{u} \times d]) & \mapsto \left(\eta_c \left(d'\mathbf{u}, \frac{\mathbf{p}-d'\mathfrak{h}(\mathbf{u})}{\left\|\mathbf{p}-d'\mathfrak{h}(\mathbf{u})\right\|}\right), [\mathbf{u} \times d']\right). \end{cases}$$

We define the map $F^c : \mathcal{F}_c \to \mathbf{F}$ analogously: let

$$F_1^c := \eta \circ G^c : \mathcal{F}_c \to \mathbf{P}$$

and let

$$F_2^c := T^{-1} \circ \mathrm{Proj}_2^c : \mathcal{F}_c \to \mathbf{C},$$

where

$$\mathrm{Proj}_2^c : \mathcal{F}_c \to \mathcal{C}_c : (\mathbf{q}, [\mathbf{x} \times d]) \to [\mathbf{x} \times d]$$

is the projection map. Then $F^c := F_1 \times F_2$ is the continuous map $\mathcal{F}_c \to \mathbf{F}$:

$$\begin{cases} (\mathbf{q}, [\mathbf{u} \times 0]) & \mapsto (\eta(\mathbf{0}, \mathbf{q}), [\mathbf{u} \times 0])\,, \\ (\mathbf{q}, [\mathbf{u} \times d]) & \mapsto \left(\eta \left(d\mathfrak{h}(\mathbf{u}), \frac{\mathbf{p}-d\mathbf{u}}{\|\mathbf{p}-d\mathbf{u}\|}\right), [\mathbf{u} \times \mathfrak{f}(\mathbf{u})d]\right). \end{cases}$$

By construction, F and F^c are inverses, so we obtain that F is a homeomorphism and Theorem 5.2.1 is proven.

5.2.2 Classification of the Flag Space

We proceed to describe the homeomorphic type of the flag space of an embeddable spherical circle plane. By Theorem 5.2.1, it suffices to do so for the classical case.

Theorem 5.2.4. *The flag space $\mathcal{F}_c$ of the classical flat Möbius plane, and hence the flag space $\mathbf{F}$ of an embeddable spherical circle plane, is homeomorphic to the* *space*

$$(\mathbb{S}_1^2 \times \mathbb{P}_2\mathbb{R})\backslash\{(\mathbf{w}, [\mathbf{w}]) : \mathbf{w} \in \mathbb{S}_1^2\}.$$

Proof. Let $H_1 : \mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o} \to \mathbb{P}_2\mathbb{R}$ be the map $[\mathbf{x} \times d] \mapsto [\mathbf{x}]$. If $\mathrm{Proj} : \mathbf{x} \times d \mapsto \mathbf{x}$ is the projection map $\mathbb{R}^4 \to \mathbb{R}^3$, we have:

$$\begin{tikzcd} m\mathbf{x} \times md \arrow[r, mapsto, "\mathrm{Proj}"] \arrow[d, mapsto, "\pi_3"'] & m\mathbf{x} \arrow[d, mapsto, "\pi_2"] \\ {[\mathbf{x} \times d]} \arrow[r, mapsto, "H_1"'] & {[m\mathbf{x}] = [\mathbf{x}]} \end{tikzcd}$$

for $m \neq 0$, which by Theorem A.1.2 implies that H_1 is continuous.

Define the map

$$H : \mathcal{F}_c \to (\mathbb{S}_1^2 \times \mathbb{P}_2\mathbb{R})\backslash\{(\mathbf{w}, [\mathbf{w}]) : \mathbf{x} \in \mathbb{S}_1^2\}$$

by $H = \mathrm{id}_{\mathbb{S}_1^2} \times H_1 : (\mathbf{p}, [\mathbf{x} \times d]) \mapsto (\mathbf{p}, [\mathbf{x}])$, where $\mathbf{p} \cdot \mathbf{x} = d$.

Let us show that H is surjective. Let

$$(\mathbf{p}, [\mathbf{x}]) \in (\mathbb{S}_1^2 \times \mathbb{P}_2\mathbb{R})\backslash\{(\mathbf{w}, [\mathbf{w}]) : \mathbf{w} \in \mathbb{S}_1^2\};$$

put $\mathbf{u} := \frac{\mathbf{x}}{||\mathbf{x}||} \in \mathbb{S}_1^2$, so $(\mathbf{p}, [\mathbf{x}]) = (\mathbf{p}, [\mathbf{u}])$.

If θ is the angle between $\mathbf{p}$ and $\mathbf{u}$, then, as $[\mathbf{p}] \neq [\mathbf{u}]$, we have that $|\cos\theta| < 1$. Hence

$$0 \leq |\mathbf{p} \cdot \mathbf{u}| = ||\mathbf{p}||||\mathbf{u}||| \cos\theta| = |\cos\theta| < 1.$$

If $\mathbf{p} \cdot \mathbf{u} = 0$ then $(\mathbf{p}, [\mathbf{u}]) = H(\mathbf{p}, [\mathbf{u} \times 0])$. On the other hand, if $0 < |\mathbf{p} \cdot \mathbf{u}| < 1$, then

$$(\mathbf{p}, [\mathbf{u}]) = H(\mathbf{p}, [\mathbf{v} \times d]),$$

where $d = |\mathbf{p}\cdot\mathbf{u}| > 0$ and $\mathbf{v} = \frac{\mathbf{p}\cdot\mathbf{u}}{|\mathbf{p}\cdot\mathbf{u}|}\mathbf{u}$. Since

$$\mathbf{p}\cdot\mathbf{v} = \mathbf{p}\cdot\left(\tfrac{\mathbf{p}\cdot\mathbf{u}}{|\mathbf{p}\cdot\mathbf{u}|}\mathbf{u}\right) = |\mathbf{p}\cdot\mathbf{u}| = d$$

and $0 < d < 1$, we see that $(\mathbf{p}, [\mathbf{v}\times d]) \in \mathcal{F}_c$. Whence H is surjective.

Now, let

$$H_2 : (\mathbb{S}^2_1 \times \mathbb{P}_2\mathbb{R})\backslash\{(\mathbf{w}, [\mathbf{w}]) : \mathbf{w}\in\mathbb{S}^2_1\} \to \mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$$

be the map $(\mathbf{p}, [\mathbf{x}]) \mapsto [\mathbf{u}\times(\mathbf{p}\cdot\mathbf{u})]$, where $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$. Since $\mathbb{S}^2_1$ is locally compact, by Theorem A.1.4 we have that $\mathrm{id}_{\mathbb{S}^2_1}\times\pi_2$ is a quotient map. Let h be the continuous map $(\mathbf{p}, \mathbf{x}) \mapsto \frac{\mathbf{x}}{||\mathbf{x}||}\times\left(\mathbf{p}\cdot\frac{\mathbf{x}}{||\mathbf{x}||}\right)$. From the commutative diagram:

$$\begin{tikzcd}
(\mathbf{p},\mathbf{x}) \arrow[r, mapsto, "h"] \arrow[d, mapsto, "\mathrm{id}_{\mathbb{S}^2_1}\times\pi_2"'] & \mathbf{u}\times(\mathbf{p}\cdot\mathbf{u}) \arrow[d, mapsto, "\pi_3"] \\
(\mathbf{p},[\mathbf{u}]) \arrow[r, mapsto, "H_2"'] & [\mathbf{u}\times(\mathbf{p}\cdot\mathbf{u})]
\end{tikzcd}$$

where $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$, we obtain that H_2 is continuous.

We claim that

$$\mathrm{id}_{\mathbb{S}^2_1}\times H_2 : (\mathbf{p}, [\mathbf{x}]) \mapsto (\mathbf{p}, [\mathbf{u}\times(\mathbf{p}\cdot\mathbf{u})])\,,$$

where $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$, is the inverse mapping of H.

Let $(\mathbf{p}, [\mathbf{u}\times d]) \in \mathcal{F}_c$, with $\mathbf{u}\in\mathbb{S}^2_1$; so $0 \le \mathbf{p}\cdot\mathbf{u} = d < 1$. Then

$$(\mathbf{p}, [\mathbf{u}\times d]) \overset{H}{\longmapsto} (\mathbf{p}, [\mathbf{u}]) \xmapsto{\mathrm{id}_{\mathbb{S}^2_1}\times H_2} (\mathbf{p}, [\mathbf{u}\times(\mathbf{p}\cdot\mathbf{u})]) = (\mathbf{p}, [\mathbf{u}\times d]).$$

Conversely, let $(\mathbf{p}, [\mathbf{u}]) \in (\mathbb{S}^2_1 \times \mathbb{P}_2\mathbb{R})\backslash\{(\mathbf{w}, [\mathbf{w}]) : \mathbf{w}\in\mathbb{S}^2_1\}$, with $\mathbf{u}\in\mathbb{S}^2_1$. Then

$$(\mathbf{p}, [\mathbf{u}]) \xmapsto{\mathrm{id}_{\mathbb{S}^2_1}\times H_2} (\mathbf{p}, [\mathbf{u}\times(\mathbf{p}\cdot\mathbf{u})]) \overset{H}{\longmapsto} (\mathbf{p}, [\mathbf{u}]).$$

Whence the continuous maps H and $\mathrm{id}_{\mathbb{S}^2_1} \times H_2$ are inverses, so H is a homeomorphism and Theorem 5.2.4 is proven. $\square$

Chapter 6

Isotopy Equivalence of SCPs

Informally, in general topology two spaces are said to be isotopy equivalent if one can be continuously deformed into the other in a completely reversible manner. In the paper [7], Rosehr defines the notion of an isotopy equivalence between so-called *stable planes*, which are point-line geometries with point and line sets carrying topologies satisfying certain properties[1]. In this chapter we first modify Rosehr's formulation and define isotopy equivalence between spherical circle planes. We then show that, with this definition, embeddable spherical circle planes are isotopy equivalent to the classical flat Möbius plane.

Since we wish an isotopy to be a continuous map into a certain collection of subsets of $\mathbb{S}^2_1$, we introduce a modified version of the "Hausdorff topology" of [8], called the Hausdorff-convergence topology, in order to topologise this set.

[1] For more information about topological constructions on point-line geometries the reader is referred to [8].

6.1 The Hausdorff-convergence Topology

Let $\mathcal{H}(\mathbb{S}^2_1)$ denote the set of all subsets of $\mathbb{S}^2_1$ that are homeomorphic to $\mathbb{S}^1_1$. We topologise $\mathcal{H}(\mathbb{S}^2_1)$ as follows. Let (M_n) be a sequence in $\mathcal{H}(\mathbb{S}^2_1)$. If $(\mathbf{p}_n)$ is a sequence in $\mathbb{S}^2_1$ such that $\mathbf{p}_n \in M_n$ for each n, we write $(\mathbf{p}_n) \in (M_n)$. Let $\lim_n \inf M_n$ be the set of all limit points of convergent sequences $(\mathbf{p}_n) \in (M_n)$, and let $\lim_n \sup M_n$ be the set of all accumulation points of all sequences $(\mathbf{q}_n) \in (M_n)$. We say that (M_n) converges to $M \in \mathcal{H}(\mathbb{S}^2_1)$ in the Hausdorff-convergence sense, and write $(M_n) \xrightarrow{HC} M$, if

$$\lim_n \inf M_n = M = \lim_n \sup M_n.$$

Note that since a limit point of a convergent sequence is an accumulation point of that sequence, we already have that $\lim_n \inf M_n \subseteq \lim_n \sup M_n$; hence the above equality holds if

$$\lim_n \sup M_n \subseteq M \subseteq \lim_n \inf M_n. \tag{$\star$}$$

Define the *Hausdorff-convergence topology* H on $\mathcal{H}(\mathbb{S}^2_1)$ as follows. A set $A \subseteq \mathcal{H}(\mathbb{S}^2_1)$ is closed with respect to H if and only if whenever a sequence (M_n) in A converges to M in the Hausdorff-convergence sense, we have that $M \in A$. The open subsets of $\mathcal{H}(\mathbb{S}^2_1)$ with respect to H are precisely the complements of the closed subsets. That H is indeed a topology on $\mathcal{H}(\mathbb{S}^2_1)$ is shown in the Appendix (Theorem A.2.2).

To prove continuity results with respect to this topology we shall make use of the following proposition (the proof of which is left to the Appendix; see Proposition A.2.3):

Proposition 6.1.1. *Let X be a metric space and endow $\mathcal{H}(\mathbb{S}^2)$ with the Hausdorff-*

convergence topology. Suppose that $g : X \to \mathcal{H}(\mathbb{S}^2)$ is map such that whenever there is a sequence (y_n) in X converging to a point y, we have $g(y_n) \xrightarrow{HC} g(y)$. Then g is continuous.

As a first step towards justifying this topology, we establish that convergence of planes in $\mathbb{R}^3$ under their identification with points in $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$ implies convergence in the Hausdorff-convergence sense. Recall that $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$ is the disjoint union of the subsets $\mathcal{Y}_1 = \{[\mathbf{u} \times 0] : \mathbf{u} \in \mathbb{S}^2_1\}$ and $\mathcal{Y}_2 = \{[\mathbf{u} \times d] : \mathbf{u} \in \mathbb{S}^2_1,\, d > 0\}$.

Theorem 6.1.2. *A sequence (Y_n) of planes in $\mathbb{R}^3$ converges to a plane Y with respect to the $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$ topology if and only if (Y_n) converges to Y in the Hausdorff-convergence sense.*

Proof. Let (Y_n) be a sequence in $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$ converging to an element Y. We verify that $(\star)$ holds.

Consider first the case that $Y \in \mathcal{Y}_2$. Since $\mathcal{Y}_2$ is open, $Y_n \in \mathcal{Y}_2$ for sufficiently large n. Put $Y_n = [\mathbf{u}_n \times d_n]$, with $\mathbf{u}_n \in \mathbb{S}^2_1$ and $d_n > 0$ for each n; let $Y = [\mathbf{u} \times d]$, with $\mathbf{u} \in \mathbb{S}^2_1$, $d > 0$. By the homeomorphism

$$\mathcal{Y}_2 \to \mathbb{S}^2_1 \times \mathbb{R}_{>0} : [\mathbf{x} \times m] \mapsto \tfrac{|m|}{m} \tfrac{\mathbf{x}}{||\mathbf{x}||} \times \tfrac{|m|}{||\mathbf{x}||},$$

since $Y_n \to Y$ we have that $\mathbf{u}_n \to \mathbf{u}$ and $d_n \to d$.

We show that $\lim_n \sup Y_n \subseteq Y$. Let $(\mathbf{p}_n) \in (Y_n)$; so $\mathbf{p}_n \cdot \mathbf{u}_n = d_n$ for each n. Suppose that $\mathbf{p}$ is the limit of some subsequence $(\mathbf{p}_{n_i})$ of $(\mathbf{p}_n)$. Then, by the continuity of the dot product, we have

$$0 = (\mathbf{p}_{n_i} - d_{n_i}\mathbf{u}_{n_i}) \cdot \mathbf{u}_{n_i} \xrightarrow{i \to \infty} (\mathbf{p} - d\mathbf{u}) \cdot \mathbf{u};$$

hence $\mathbf{p} \cdot \mathbf{u} = d$ and so $\mathbf{p} \in Y$.

We now show that $Y \subseteq \lim_n \inf Y_n$. Let $\mathbf{p} \in Y$, so $\mathbf{p} \cdot \mathbf{u} = d$. Define the sequence $(\mathbf{p}_n)$ by

$$\mathbf{p}_n = \mathbf{p} - (\mathbf{p} \cdot \mathbf{u}_n - d_n)\mathbf{u}_n$$

for each n. Then

$$\mathbf{p}_n \cdot \mathbf{u}_n = \mathbf{p} \cdot \mathbf{u}_n - (\mathbf{p} \cdot \mathbf{u}_n - d_n) = d_n$$

for each n, so $(\mathbf{p}_n) \in (Y_n)$. Moreover,

$$\mathbf{p}_n = \mathbf{p} - (\mathbf{p} \cdot \mathbf{u}_n - d_n)\mathbf{u}_n \to \mathbf{p} - (\mathbf{p} \cdot \mathbf{u} - d)\mathbf{u} = \mathbf{p},$$

as $n \to \infty$, which implies that $\mathbf{p} \in \lim_n \inf Y_n$.

Now consider consider the case when $Y \in \mathcal{Y}_1$; put $Y = [\mathbf{u} \times 0] \in \mathcal{Y}_1$, where $\mathbf{u} \in \mathbb{S}^2_1$.

If $Y_n \in \mathcal{Y}_1$ for only finitely many n, then by removing these elements from the sequence and relabelling we have $Y_n \in \mathcal{Y}_2$ for all n. Put $Y_n = [\mathbf{u}_n \times d_n]$, with $\mathbf{u}_n \in \mathbb{S}^2_1$ and $d_n > 0$ for each n. Then

$$[\mathbf{u}_n \times d_n] = [\mathbf{x}_n \times 1] \to [\mathbf{x} \times 0]$$

and so by Lemma 4.3.2 we have that $\mathbf{u}_n \to \mathbf{u}$ and $d_n \to 0$.

To show that $\lim_n \sup Y_n \subseteq Y$, let $(\mathbf{p}_n) \in (Y_n)$ and suppose that $\mathbf{p}$ is the limit of some subsequence $(\mathbf{p}_{n_i})$ of $(\mathbf{p}_n)$. We have

$$0 = \mathbf{p}_{n_i} \cdot \mathbf{u}_{n_i} - d_{n_i} \xrightarrow{i \to \infty} \mathbf{p} \cdot \mathbf{u} - 0 = \mathbf{p} \cdot \mathbf{u}$$

and hence $\mathbf{p} \in Y$.

We now show that $Y \subseteq \lim_n \inf Y_n$. Let $\mathbf{p} \in Y$, so $\mathbf{p} \cdot \mathbf{u} = 0$. Define the sequence $(\mathbf{p}_n)$ by

$$\mathbf{p}_n = \mathbf{p} - (\mathbf{p} \cdot \mathbf{u}_n - d_n)\mathbf{u}_n$$

for each n. Then

$$\mathbf{p}_n \cdot \mathbf{u}_n = \mathbf{p} \cdot \mathbf{u}_n - (\mathbf{p} \cdot \mathbf{u}_n - d_n) = d_n$$

for each n, so $(\mathbf{p}_n) \in (Y_n)$. Moreover, we have

$$\mathbf{p}_n = \mathbf{p} - (\mathbf{p} \cdot \mathbf{u}_n - d_n)\mathbf{u}_n \xrightarrow{n\to\infty} \mathbf{p} - (\mathbf{p} \cdot \mathbf{u})\mathbf{u} = \mathbf{p}$$

since $\mathbf{p} \cdot \mathbf{u} = 0$; whence $\mathbf{p} \in \lim_n \inf Y_n$.

Now assume that $Y_n \in \mathcal{Y}_1$ for infinitely many n.

Suppose there is an $N > 0$ such that $Y_n \in \mathcal{Y}_1$ for all $n > N$. Put $Y_n = [\mathbf{u}_n \times 0]$, with $\mathbf{u}_n \in \mathbb{S}^2_1$ for each n. Then $[\mathbf{u}_n \times 0] \to [\mathbf{u} \times 0]$, so $[\mathbf{u}_n] \to [\mathbf{u}]$.

To show that $\lim_n \sup Y_n \subseteq Y$, let $(\mathbf{p}_n) \in (Y_n)$ and suppose that $\mathbf{p}$ is the limit of some subsequence $(\mathbf{p}_{n_i})$ of $(\mathbf{p}_n)$. Then

$$0 = |\mathbf{p}_{n_i} \cdot \mathbf{u}_{n_i}| \xrightarrow{i\to\infty} |\mathbf{p} \cdot \mathbf{u}|,$$

so $\mathbf{p} \cdot \mathbf{u} = 0$ and $\mathbf{p} \in Y$.

To show that $Y \subseteq \lim_n \inf Y_n$, let $\mathbf{p} \in Y$ and so $\mathbf{p} \cdot \mathbf{u} = 0$. Define the sequence $(\mathbf{p}_n)$ by

$$\mathbf{p}_n = \mathbf{p} - (\mathbf{p} \cdot \mathbf{u}_n)\mathbf{u}_n$$

for each n. Then

$$\mathbf{p}_n \cdot \mathbf{u}_n = \mathbf{p} \cdot \mathbf{u}_n - \mathbf{p} \cdot \mathbf{u}_n = 0$$

for each n, so $(\mathbf{p}_n) \in (Y_n)$. Moreover, since $|\mathbf{p} \cdot \mathbf{u}_n| \xrightarrow{n\to\infty} |\mathbf{p} \cdot \mathbf{u}| = 0$, we have that $\mathbf{p} \cdot \mathbf{u}_n \to 0$, and hence

$$\mathbf{p}_n = \mathbf{p} - (\mathbf{p} \cdot \mathbf{u}_n)\mathbf{u}_n \to \mathbf{p} - 0\mathbf{u} = \mathbf{p},$$

as $n \to \infty$. This implies that $\mathbf{p} \in \lim_n \inf Y_n$.

On the other hand, let $(Y_m)_{m \in N_1}$ and $(Y_{m'})_{m' \in N_2}$ be infinite subsequences of (Y_n) corresponding to the elements in $\mathcal{Y}_1$ and $\mathcal{Y}_2$, respectively, where N_1 and N_2 partition $\mathbb{N}$. By the cases "$Y \in \mathcal{Y}_1$ and $Y_n \in \mathcal{Y}_1$ for all n" and "$Y \in \mathcal{Y}_1$ and $Y_n \in \mathcal{Y}_2$ for all n" just considered, respectively, it is apparent that $\lim_m \sup Y_m = Y = \lim_m \inf Y_m$ and $\lim_{m'} \sup Y_{m'} = Y = \lim_{m'} \inf Y_{m'}$.

If $\mathbf{p}$ is the limit of some subsequence $(\mathbf{p}_{n_i})$ of a sequence $(\mathbf{p}_n) \in (Y_n)$, then $\mathbf{p}$ is the limit of a subsequence of $(\mathbf{p}_n)$ such that either each term lies in a subsequence of (Y_m), or each term lies in a subsequence of $(Y_{m'})$, so $\mathbf{p} \in Y$; hence $\lim_n \sup Y_n \subseteq Y$.

If $\mathbf{p} \in Y$, then $\mathbf{p}$ is the limit of some sequences $(\mathbf{p}_m)$, $(\mathbf{p}_{m'})$ in (Y_m), $(Y_{m'})$, respectively, and hence is the limit of the sequence $(\mathbf{p}_n)$ in (Y_n) given by

$$\mathbf{p}_n = \begin{cases} \mathbf{p}_m & \text{if } n = m \in N_1, \\ \mathbf{p}_{m'} & \text{if } n = m' \in N_2, \end{cases}$$

which converges to $\mathbf{p}$. Thus $Y \subseteq \lim_n \inf Y_n$.

This completes the proof that $Y_n \xrightarrow{HC} y$.

Conversely, suppose that $(Y_n) = ([\mathbf{x}_n \times d_n])$ is a sequence of planes converging with respect to H to a plane $Y = [\mathbf{x} \times d]$. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three non-collinear points on Y. Since $Y \subseteq \lim_n \inf Y_n$, the points $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are limits of some sequences $(\mathbf{u}_n)$, $(\mathbf{v}_n)$, $(\mathbf{w}_n) \in (Y_n)$, respectively. Note that if $\mathbf{u}_n$, $\mathbf{v}_n$ and $\mathbf{w}_n$ are not distinct for infinitely many n, then the corresponding subsequences converge to non-distinct points — a contradiction. Suppose they have subsequences $(\mathbf{u}_{n_k})$,

$(\mathbf{v}_{n_k})$ and $(\mathbf{w}_{n_k})$ such that $\mathbf{u}_{n_k}$, $\mathbf{v}_{n_k}$ and $\mathbf{w}_{n_k}$ are distinct and collinear for each k. Then

$$L(\mathbf{u}_{n_k}, [\mathbf{u}_{n_k} - \mathbf{v}_{n_k}]) = L(\mathbf{u}_{n_k}, [\mathbf{u}_{n_k} - \mathbf{w}_{n_k}])$$

for each k. By Lemma 3.4.10, we have

$$L(\mathbf{u}_{n_k}, [\mathbf{u}_{n_k} - \mathbf{v}_{n_k}]) \xrightarrow{k\to\infty} L(\mathbf{u}, [\mathbf{u} - \mathbf{v}])$$

and

$$L(\mathbf{u}_{n_k}, [\mathbf{u}_{n_k} - \mathbf{w}_{n_k}]) \xrightarrow{k\to\infty} L(\mathbf{u}, [\mathbf{u} - \mathbf{w}])$$

with respect to the (metric) topology on $\mathcal{L}$, so

$$L(\mathbf{u}, [\mathbf{u} - \mathbf{v}]) = L(\mathbf{v}, [\mathbf{v} - \mathbf{w}]).$$

This implies that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are collinear — a contradiction.

Thus, by removing finitely many terms of (Y_n) if necessary and by relabelling, we may assume that $\mathbf{u}_n$, $\mathbf{v}_n$, and $\mathbf{w}_n$ are not collinear for all n. By the continuity of the joining map (cf. Proposition 3.3.1) we therefore have

$$Y_n = \alpha(\mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n) \to \alpha(\mathbf{u}, \mathbf{v}, \mathbf{w}) = Y$$

as $n \to \infty$ with respect to the topology of $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$. $\square$

6.2 Isotopy Equivalence Defined

We now give a definition of isotopy of spherical circle planes.

Definition Let $(\mathcal{P}_1, \mathcal{C}_1)$ and $(\mathcal{P}_2, \mathcal{C}_2)$ be spherical circle planes. A map $G : [0,1] \times \mathcal{C}_1 \to \mathcal{H}(\mathcal{P}_2)$ is called an *isotopy* from $(\mathcal{P}_1, \mathcal{C}_1)$ to $(\mathcal{P}_2, \mathcal{C}_2)$ if the following conditions hold. Let G_t denote the map $G_t(C) = G(t, C)$.

(I1) There is a bijection $\kappa : \mathcal{P}_1 \to \mathcal{P}_2$ such that[2]

$$G_0(C) = \kappa[C] := \bigcup_{\mathbf{x} \in C} \{\kappa(\mathbf{x})\}$$

for each $C \in \mathcal{C}_1$.

(I2) $G_1(\mathcal{C}_1) = \mathcal{C}_2$.

(I3) $(\mathcal{P}_2, G_t(\mathcal{C}_1))$ is a spherical circle plane for all $t \in [0,1]$.

(I4) G is continuous.

(I5) $G_t : \mathcal{C}_1 \to G_t(\mathcal{C}_1)$ is a homeomorphism for all $t \in [0,1]$.

The spherical circles planes $(\mathcal{P}_1, \mathcal{C}_1)$ and $(\mathcal{P}_2, \mathcal{C}_2)$ are then said to be *isotopy equivalent*.

6.3 An Isotopy of ESCPs

We proceed to construct an isotopy from an embeddable spherical circle plane $(\mathbf{P}, \mathbf{C})$ to the classical flat Möbius plane $(\mathbb{S}^2_1, \mathcal{C}_c)$.

6.3.1 The Intermediate ESCPs

Recall that we choose the origin to be a point in $\mathbf{B}$; without loss of generality assume that $\mathbb{S}^2_1$ is strictly contained in $\mathbf{B}$.

For each $t \in [0,1]$, let $m_t : \overline{\mathbf{B}} \to m_t(\overline{\mathbf{B}}) \subseteq \mathbb{R}^3$ be the map defined by

$$m_t(\mathbf{x}) = (t + (1-t)||\mathbf{x}||)\tfrac{\mathbf{x}}{||\mathbf{x}||}.$$

For $t \in [0,1)$, we have

$$m_t^{-1}(\mathbf{x}) = \frac{||\mathbf{x}|| - t}{1-t}\frac{\mathbf{x}}{||\mathbf{x}||}$$

[2] We denote the image of a subset $A \subseteq X$ under a map $f : X \to Y$ by $f[A]$.

(one can readily verify that the maps $k \mapsto t+(1-t)k$ and $k \mapsto \frac{k-t}{1-t}$ are inverses).

For $t = 1$, $m_1(\mathbf{x}) = \frac{\mathbf{x}}{||\mathbf{x}||}$ is a homeomorphism since $\overline{\mathbf{B}}$ is compact and convex (cf. Theorem A.1.14). Whence m_t is a homeomorphism for each t.

Let $\mathbf{B}_t := m_t(\mathbf{B})$; then

$$\mathbf{P}_t := \partial \mathbf{B}_t = \partial(m_t(\mathbf{B})) = m_t(\partial \mathbf{B}) = m_t(\mathbf{P}).$$

That is,

$$\mathbf{P}_t = \left\{(t + (1-t)||\mathbf{x}||)\tfrac{\mathbf{x}}{||\mathbf{x}||} : \mathbf{x} \in \mathbf{P}\right\}.$$

We first show that the strict convexity of $\mathbf{P}$ implies that $\mathbf{P}_t$ is strictly convex, thus allowing us to define the embeddable spherical circle plane $(\mathbf{P}_t, \mathbf{C}_t)$ on $\mathbf{P}_t$.

Proposition 6.3.1. *The subset $\mathbf{P}_t$ of $\mathbb{R}^3$ is strictly convex for all $t \in [0,1]$.*

Proof. Obviously $\mathbf{P}_0 = \mathbf{P}$ and $\mathbf{P}_1 = \mathbb{S}^2_1$ are strictly convex; let $t \in (0,1)$ and let $\mathbf{x}_t$, $\mathbf{y}_t$ and $\mathbf{z}_t$ be distinct, collinear points of $\mathbf{P}_t$. Put $\mathbf{x} = m_t^{-1}(\mathbf{x}_t)$, $\mathbf{y} = m_t^{-1}(\mathbf{y}_t)$ and $\mathbf{z} = m_t^{-1}(\mathbf{z}_t) \in \mathbf{P}$.

By projecting onto the plane containing the points $\mathbf{0}, \mathbf{x}_t$ and $\mathbf{y}_t$, and rotating the coordinate axes, we may depict the open line segment between $\mathbf{x}_t$ and $\mathbf{y}_t$ as in Figure 6.3.1.1; if $\mathbf{w}_1$, $\mathbf{w}_2$ lie in the (u,v)-plane and satisfy $||\mathbf{w}_1|| < ||\mathbf{w}_2||$, then their preimages $\mathbf{w}'_1$, $\mathbf{w}'_2$ under this projection satisfy $||\mathbf{w}'_1|| < ||\mathbf{w}'_2||$ in $\mathbb{R}^3$. Let $v = k$ represent the line through $\mathbf{x}_t$, $\mathbf{y}_t$ and $\mathbf{z}_t$ in the (u,v)-plane .

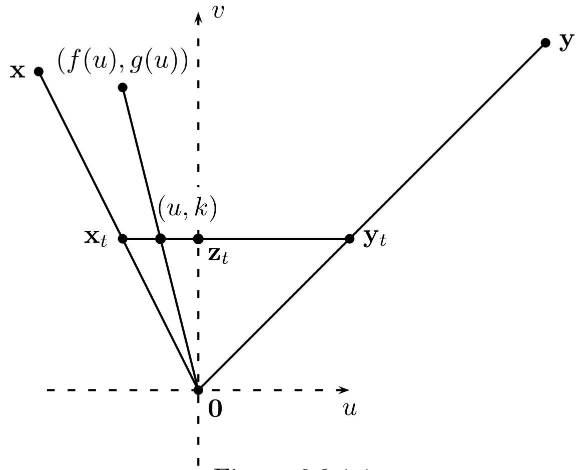


Figure 6.3.1.1

The image of a point (u, k) in the (u, v)-plane corresponding to the image of a point in $]\mathbf{x}_t, \mathbf{y}_t[$ under m_t^{-1} is given by the parametric curve $\Gamma(u) = (f(u), g(u))$, where

$$g(u) = \frac{\sqrt{u^2 + k^2} - t}{1 - t} \frac{k}{\sqrt{u^2 + k^2}}.$$

One calculates that the second derivative of g at $u = 0$ is

$$g''(0) = \frac{t}{(1 - t)k^2} > 0,$$

so the image of a neighbourhood of $\mathbf{z}_t$ in $]\mathbf{x}_t, \mathbf{y}_t[$ under m_t^{-1} is strictly concave. Thus the line segments $]\mathbf{x}, \mathbf{z}[$ and $]\mathbf{z}, \mathbf{y}[$ lie inside the triangle formed by $\mathbf{0}$, $\mathbf{x}$ and $\mathbf{y}$; see Figure 6.3.1.2. This contradicts the strict convexity of $\mathbf{P}$, so we deduce that $\mathbf{P}_t$ is also strictly convex.

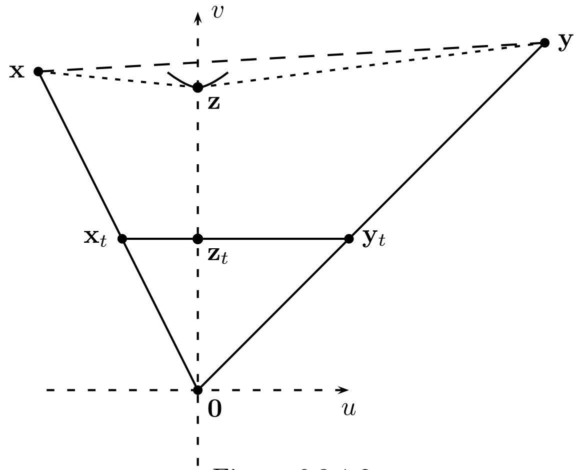


Figure 6.3.1.2 $\square$

Let $t \in [0,1]$. Following Proposition 6.3.1 we may thus define the embeddable spherical plane $(\mathbf{P}_t, \mathbf{C}_t)$ on $\mathbf{P}_t$; define the distance map $\mathfrak{f}_t := \mathfrak{f}_{\mathbf{P}_t}$. Let us give an explicit description of $\mathfrak{f}_t$ (which is necessarily continuous) in terms of the distance map $\mathfrak{f}$ and maximal map $\mathfrak{h}$ associated with $\mathbf{P}$. Recalling that $H_{\mathbf{y}}$ is the particular open half-space determined by $[\mathbf{y} \times 0]$ given by

$$H_{\mathbf{y}} = \{\mathbf{z} \in \mathbb{R}^3 : \mathbf{y} \cdot \mathbf{z} > 0\}$$

and that each $\mathbf{p}_t \in \mathbf{P}_t$ is of the form $\big(t + (1-t)||\mathbf{p}||\big)\frac{\mathbf{p}}{||\mathbf{p}||}$ for some $\mathbf{p} \in \mathbf{P}$, we obtain:

$$\begin{aligned}
\mathfrak{f}_t(\mathbf{u}) &= \max_{\mathbf{p}_t \in \mathbf{P}_t \cap H_{\mathbf{u}}} \mathbf{p}_t \cdot \mathbf{u} \\
&= \max_{\mathbf{p} \in \mathbf{P} \cap H_{\mathbf{u}}} \big(t + (1-t)||\mathbf{p}||\big)\tfrac{\mathbf{p}}{||\mathbf{p}||} \cdot \mathbf{u} \\
&= \big(t + (1-t)||\mathfrak{h}(\mathbf{u})||\big)\tfrac{\mathfrak{h}(\mathbf{u})}{||\mathfrak{h}(\mathbf{u})||} \cdot \mathbf{u} \\
&= \left(\tfrac{t}{||\mathfrak{h}(\mathbf{u})||} + (1-t)\right)\mathfrak{f}(\mathbf{u}).
\end{aligned}$$

With the distance map for $\mathbf{P}_t$ now at hand we may characterise $\mathbf{C}_t$ as:

$$\mathbf{C}_t = \{[\mathbf{u} \times 0] : \mathbf{u} \in \mathbb{S}_1^2\} \cup \{[\mathbf{u} \times d] : \mathbf{u} \in \mathbb{S}_1^2, 0 < d < \mathfrak{f}_t(\mathbf{u})\}.$$

We now describe a homeomorphism from $\mathbf{C}$ to $\mathbf{C}_t$, in direct analogy to our construction of a homeomorphism from $\mathbf{C}$ to $\mathcal{C}_c$ in Chapter 5.

Define $\zeta_t : \mathbb{R}^3 \backslash \mathbf{0} \to \mathbb{R}^3 \backslash \mathbf{0}$ by

$$\begin{aligned}\zeta_t(\mathbf{x}) &= \frac{\mathfrak{f}_t(\mathbf{u})}{\mathfrak{f}(\mathbf{u})}\mathbf{x} \\ &= \left(\tfrac{t}{||\mathfrak{h}(\mathbf{u})||} + (1-t)\right)\mathbf{x},\end{aligned}$$

where $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$. Then ζ_t is continuous and has continuous inverse $\mathbf{x} \mapsto \frac{\mathfrak{f}(\mathbf{u})}{\mathfrak{f}_t(\mathbf{u})}\mathbf{x}$. Furthermore,

$$\frac{\zeta_t(\mathbf{x})}{||\zeta_t(\mathbf{x})||} = \frac{\mathbf{x}}{||\mathbf{x}||} = \frac{\zeta_t^{-1}(\mathbf{x})}{||\zeta_t^{-1}(\mathbf{x})||}$$

for each $\mathbf{x} \in \mathbb{R}^3 \backslash \mathbf{0}$, and for any sequence $(\mathbf{x}_n)$ in $\mathbb{R}^3 \backslash \mathbf{0}$ such that $\frac{1}{||\mathbf{x}_n||} \to 0$:

$$\frac{1}{||\zeta_t(\mathbf{x}_n)||} = \frac{\mathfrak{f}(\mathbf{u}_n)}{\mathfrak{f}_t(\mathbf{u}_n)} \frac{1}{||\mathbf{x}_n||} \to 0 \ , \ \frac{1}{||\zeta_t^{-1}(\mathbf{x}_n)||} = \frac{\mathfrak{f}_t(\mathbf{u}_n)}{\mathfrak{f}(\mathbf{u}_n)} \frac{1}{||\mathbf{x}_n||} \to 0,$$

where $(\mathbf{u}_n)$ is the sequence given by $\mathbf{u}_n = \frac{\mathbf{x}_n}{||\mathbf{x}_n||}$ for each n, and where we have used that $(\mathfrak{f}(\mathbf{u}_n))$ and $(\mathfrak{f}_t(\mathbf{u}_n))$ are bounded above and away from 0. Let $\Upsilon_t : \mathbf{C} \to \mathbf{C}_t$ be the map defined on $\mathcal{Y}_1 \cap \mathbf{C}$ and $\mathcal{Y}_2 \cap \mathbf{C}$ by

$$\begin{aligned}&[\mathbf{x} \times 0] \mapsto [\zeta_t(\mathbf{x}) \times 0] = [\mathbf{x} \times 0], \\ &[\mathbf{x} \times 1] \mapsto [\zeta_t(\mathbf{x}) \times 1],\end{aligned}$$

respectively. Note that Υ_t does indeed map into $\mathbf{C}_t$ since, putting $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$, we have that $\frac{1}{||\mathbf{x}||} < f(\mathbf{u})$ if and only if

$$\frac{1}{||\zeta_t(\mathbf{x})||} = \frac{\mathfrak{f}(\mathbf{u})}{\mathfrak{f}_t(\mathbf{u})}\frac{1}{||\mathbf{x}||} < \mathfrak{f}_t(\mathbf{u}).$$

As Υ_t is a restriction-corestriction of a homeomorphism of the form in Theorem 4.3.5, it is a homeomorphism. Let us record this for later.

Lemma 6.3.2. *For each $t \in [0,1]$, the map $\Upsilon_t : \mathbf{C} \to \mathbf{C}_t$ is a homeomorphism.*

In particular, $\Upsilon_t(\mathbf{C}) = \mathbf{C}_t$, so we obtain a family of intermediate embeddable spherical circle planes:

Lemma 6.3.3. *For each $t \in [0,1]$, the point-circle geometry $(\mathbf{P}_t, \Upsilon_t(\mathbf{C}))$ is an embeddable spherical circle plane, and hence a spherical circle plane.*

6.3.2 The Isotopy Defined

Let $\kappa : \mathbb{R}^3 \backslash \{\mathbf{0}\} \to \mathbb{S}^2_1$ be the map $\mathbf{x} \mapsto \frac{\mathbf{x}}{||\mathbf{x}||}$, and let κ_t be the restriction of κ to $\mathbf{P}_t$. Since $\mathbf{P}_t$ is compact and convex, κ_t is a homeomorphism. Let $\mathcal{K}$ denote the induced map

$$\mathcal{K} : \bigcup_{t\in[0,1]} \mathbf{C}_t \to \mathcal{H}(\mathbb{S}^2_1),\ \mathcal{K}(C) = \kappa[C].$$

Note that $\mathcal{K}$ does indeed map into $\mathcal{H}(\mathbb{S}^2_1)$ since each $C \in \mathbf{C}_t$ is homeomorphic to $\mathbb{S}^1_1$. Let $\mathcal{K}_t$ be the restriction of $\mathcal{K}$ to $\mathbf{C}_t$; that is,

$$\mathcal{K}_t : \mathbf{C}_t \to \mathcal{K}_t(\mathbf{C}_t) \subseteq \mathcal{H}(\mathbb{S}^2_1),\ \mathcal{K}_t(C) = \kappa_t[C].$$

We are now in a position to define the desired isotopy from $(\mathbf{P}, \mathbf{C})$ to $(\mathbb{S}^2_1, \mathcal{C}_c)$. Define the map $\Xi : [0,1] \times \mathbf{C} \to \mathcal{H}(\mathbb{S}^2_1)$ by

$$\Xi(t, C) = \mathcal{K}(\Upsilon_t(C)).$$

Let Ξ_t denote the map $\Xi_t(C) = \Xi(t, C)$. Note that, for a fixed t,

$$\Xi_t(C) = \mathcal{K}(\Upsilon_t(C)) = \mathcal{K}_t(\Upsilon_t(C))$$

for each $C \in \mathbf{C}$. We proceed to verify that Ξ satisfies the conditions (I1)-(I5) of an isotopy.

(I1) Since $\mathfrak{f}_0 = \mathfrak{f}$, the map ζ_0 is the identity on $\mathbb{R}^3\backslash\mathbf{0}$ and so Υ_0 is the identity on $\mathbf{C}$. Whence, for each $C \in \mathbf{C}$:

$$\Xi_0(C) = \mathcal{K}_0(C) = \kappa_0[C],$$

where $\kappa_0 : \mathbf{P} \to \mathbb{S}^2_1$ is a bijection.

(I2) We have $\Upsilon_1(\mathbf{C}) = \mathbf{C}_1 = \mathcal{C}_c$; the map κ_1 is the identity on $\mathbf{P}_1 = \mathbb{S}^2_1$, so $\kappa_1[C'] = C'$ for each $C' \in \mathcal{C}_c$. Hence

$$\Xi_1(\mathbf{C}) = \mathcal{K}_1(\Upsilon_1(\mathbf{C})) = \mathcal{K}_1(\mathcal{C}_c) = \mathcal{C}_c.$$

(I3) Each circle of $\Xi_t(\mathbf{C})$ is of the form $\mathcal{K}_t(C_t)$ for some circle C_t in $\mathbf{C}_t$. As $(\mathbf{P}_t, \mathbf{C}_t)$ is a spherical circle plane, C_t contains at least three points in $\mathbf{P}_t$ and is homeomorphic to $\mathbb{S}^1_1$. To show that the same is true for $\mathcal{K}_t(C_t)$ we need the following.

Lemma 6.3.4. *The map $\mathcal{K}_t : \mathbf{C}_t \to \mathcal{K}_t(\mathbf{C}_t) \subseteq \mathcal{H}(\mathbb{S}^2_1)$ is a homeomorphism for each $t \in [0, 1]$.*

Proof. Let (Y_i) be a sequence in $\mathbf{C}_t$ converging to an element Y (with respect to the topology on $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$). By Theorem 6.1.2, we have $Y_n \xrightarrow{HC} Y$. We show that $(\mathcal{K}_t(Y_n)) \xrightarrow{HC} \mathcal{K}_t(Y)$.

• $\lim_n \sup \mathcal{K}_t(Y_n) \subseteq \mathcal{K}_t(Y)$: Let $(\mathbf{p}_n) \in (\mathcal{K}_t(Y_n))$. Let $\mathbf{p}$ be the limit of some convergent subsequence $(\mathbf{p}_{n_k})$ of $(\mathbf{p}_n)$. For each n, let $\mathbf{q}_n = \kappa_t^{-1}(\mathbf{p}_n)$; then

$(\mathbf{q}_n) \in (Y_n)$ and, since κ_t is a homeomorphism,

$$\mathbf{q}_{n_k} \xrightarrow{k\to\infty} \kappa_t^{-1}(\mathbf{p}) =: \mathbf{q}.$$

Hence, since $\lim_n \sup Y_n = Y$, we have $\mathbf{q} \in Y$; thus $\mathbf{p} = \kappa_t(\mathbf{q}) \in \mathcal{K}_t(Y)$. This shows that $\lim_n \sup \mathcal{K}_t(Y_n) \subseteq \mathcal{K}_t(Y)$.

• $\mathcal{K}_t(Y) \subseteq \lim_n \inf \mathcal{K}_t(Y_n)$: Let $\mathbf{p} \in \mathcal{K}_t(Y)$, so $\mathbf{p} = \kappa_t(\mathbf{q})$ for some $\mathbf{q} \in Y$. Since $Y = \lim_n \inf Y_n$, there is a sequence $(\mathbf{q}_n) \in (Y_n)$ converging to $\mathbf{q}$. Let $(\mathbf{p}_n)$ be the sequence given by

$$\mathbf{p}_n = \kappa_t(\mathbf{q}_n)$$

for each n. Then $(\mathbf{p}_n) \in (\mathcal{K}_t(Y_n))$ and

$$\mathbf{p}_n = \kappa_t(\mathbf{q}_n) \to \kappa_t(\mathbf{q}) = \mathbf{p},$$

so $\mathbf{p} \in \lim_n \inf \mathcal{K}_t(Y_n)$, as required.

We now show that $\mathcal{K}_t^{-1}$ is continuous. Let (Y_n) be a sequence in $\mathbf{C}_t$ and let $Y \in \mathbf{C}_t$ be such that the sequence $(\mathcal{K}_t(Y_n)) \xrightarrow{HC} \mathcal{K}_t(Y)$. By replacing $\mathcal{K}_t$ with $\mathcal{K}_t^{-1}$ in the above arguments used to show that $Y_n \xrightarrow{HC} Y$ implies $\mathcal{K}_t(Y_n) \xrightarrow{HC} \mathcal{K}_t(Y)$, we obtain that $\mathcal{K}_t(Y_n) \xrightarrow{HC} \mathcal{K}_t(Y)$ implies $Y_n \xrightarrow{HC} Y$. Hence, by Theorem 6.1.2, $Y_n \to Y$ in $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$. That is,

$$\mathcal{K}_t^{-1}\big(\mathcal{K}_t(Y_n)\big) = Y_n \to Y = \mathcal{K}_t^{-1}\big(\mathcal{K}_t(Y)\big)$$

in $\mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$. Whence $\mathcal{K}_t^{-1}$ is continuous. □

To complete our verification that $(\mathbb{S}_1^2, \Xi_t(\mathbf{C}))$ is a spherical circle plane, we show that any three distinct points of $\mathbb{S}_1^2$ uniquely determine a circle in $\Xi_t(\mathbf{C})$.

Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be distinct points in $\mathbb{S}_1^2$. Then $\kappa_t^{-1}(\mathbf{x})$, $\kappa_t^{-1}(\mathbf{y})$, $\kappa_t^{-1}(\mathbf{z})$ are distinct points in $\mathbf{P}_t$ and so, as $(\mathbf{P}_t, \mathbf{C}_t)$ is a spherical circle plane, there is a unique circle $C_t \in \mathbf{C}_t$ containing them. Hence $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ uniquely determine the circle $\mathcal{K}_t(C_t) \in \Xi_t(\mathbf{C})$.

(I4) We now introduce dependencies on t and show that Υ and $\mathcal{K}$ are continuous.

Lemma 6.3.5. *The map* $\Upsilon : [0,1]\times\mathbf{C} \to \Upsilon([0,1]\times\mathbf{C})$ *given by* $\Upsilon(t, C) = \Upsilon_t(C)$ *is continuous.*

Proof. Since

$$\zeta_t(\mathbf{x}) = \left(\frac{t}{||\mathfrak{h}(\mathbf{u})||} + (1-t)\right)\mathbf{x}$$

for every $\mathbf{x} \in \mathbb{R}^3\backslash\mathbf{0}$, where $\mathbf{u} = \frac{\mathbf{x}}{||\mathbf{x}||}$, we see that the map $t \mapsto \zeta_t$ is continuous; hence so is the map $(t, \mathbf{x} \times d) \mapsto \zeta_t(\mathbf{x}) \times d$. Noting that the map $\mathrm{id}_{[0,1]} \times \pi_3$ is a quotient map (since $[0,1]$ is compact; cf. Theorem A.1.4), from the commutative diagram

$$\begin{tikzcd}
(t, \mathbf{x} \times d) \arrow[r, mapsto] \arrow[d, mapsto, "\mathrm{id} \times \pi_3"'] & \zeta_t(\mathbf{x}) \times d \arrow[d, mapsto, "\pi_3"] \\
(t, [\mathbf{x} \times d]) \arrow[r, mapsto, "\Upsilon"'] & [\zeta_t(\mathbf{x}) \times d]
\end{tikzcd}$$

we obtain that Υ is continuous. $\square$

Lemma 6.3.6. *The map* $\mathcal{K} : \Upsilon([0,1]\times\mathbf{C}) \to \mathcal{H}(\mathbb{S}^2_1)$ *is continuous.*

Proof. Let $(\Upsilon(t_n, C_n))$ be a sequence in $\Upsilon([0,1] \times \mathbf{C}) \subseteq \mathbb{P}_3\mathbb{R}\backslash\mathbf{p_o}$ converging to an element $\Upsilon(t, C)$. By Proposition 6.1.2, we have $(\Upsilon(t_n, C_n)) \xrightarrow{HC} \Upsilon(t, C)$. We show that $\mathcal{K}(\Upsilon(t_n, C_n)) \xrightarrow{HC} \mathcal{K}(\Upsilon(t, C))$.

- $\lim_n \sup \mathcal{K}(\Upsilon(t_n, C_n)) \subseteq \mathcal{K}(\Upsilon(t, C))$: Let $(\mathbf{p}_n) \in (\mathcal{K}(\Upsilon(t_n, C_n)))$ and suppose that $\mathbf{p}$ is the limit of some convergent subsequence $(\mathbf{p}_{n_k})$ of $(\mathbf{p}_n)$. Let $(\mathbf{q}_n)$ be the sequence given by

$$\mathbf{q}_n = \kappa^{-1}(\mathbf{p}_n)$$

for each n; then $(\mathbf{q}_n) \in (\Upsilon(t_n, C_n))$. Since κ is a homeomorphism, the subsequence $(\mathbf{q}_{n_k})$ converges to

$$\kappa^{-1}(\mathbf{p}) =: \mathbf{q}.$$

Thus

$$\mathbf{q} \in \lim_n \sup \Upsilon(t_n, C_n) = \Upsilon(t, C),$$

and so

$$\mathbf{p} = \kappa(\mathbf{q}) \in \mathcal{K}(\Upsilon(t, C)).$$

• $\mathcal{K}(\Upsilon(t, C)) \subseteq \lim_n \inf \mathcal{K}(\Upsilon(t_n, C_n))$: Let $\mathbf{p} \in \kappa(\Upsilon(t, C))$. Then

$$\kappa^{-1}(\mathbf{p}) \in \Upsilon(t, C) = \lim_n \inf \Upsilon(t_n, C_n),$$

so $\kappa^{-1}(\mathbf{p})$ is the limit of some sequence $(\mathbf{q}_n) \in (\Upsilon(t_n, C_n))$. Hence, since κ is a homeomorphism, we have that $\mathbf{p} = \kappa(\kappa^{-1}(\mathbf{p}))$ is the limit of the sequence $(\mathbf{p}_n)$ in $\mathcal{K}(\Upsilon(t_n, C_n))$ given by $\mathbf{p}_n = \kappa(\mathbf{q}_n)$ for each n. Thus $\mathbf{p} \in \lim_n \inf \mathcal{K}(\Upsilon(t_n, C_n))$.

This shows that $\mathcal{K}(\Upsilon(t_n, C_n)) \xrightarrow{HC} \mathcal{K}(\Upsilon(t, C))$, so $\mathcal{K}$ is continuous. $\square$

The criterion (I4) is now mere formality: since

$$\begin{aligned}\Xi = \mathcal{K} \circ \Upsilon : [0,1] \times \mathbf{C} &\xrightarrow{\Upsilon} \Upsilon([0,1] \times \mathbf{C}) \xrightarrow{\mathcal{K}} \mathcal{H}(\mathbb{S}^2_1)\\ : (t, C) &\xmapsto{\Upsilon} \Upsilon(t, C)) \xmapsto{\mathcal{K}} \mathcal{K}(\Upsilon(t, C)),\end{aligned}$$

we obtain from the continuity of Υ (Lemma 6.3.5) and $\mathcal{K}$ (Lemma 6.3.6) that Ξ is continuous.

(I5) Finally, as

$$\begin{aligned}\Xi_t = \mathcal{K}_t \circ \Upsilon_t : \mathbf{C} &\xrightarrow{\Upsilon_t} \mathbf{C}_t \xrightarrow{\mathcal{K}_t} \mathcal{K}_t(\mathbf{C}_t) \subseteq \mathcal{H}(\mathbb{S}^2_1)\\ : C &\xmapsto{\Upsilon_t} \Upsilon_t(C) \xmapsto{\mathcal{K}_t} \mathcal{K}_t(\Upsilon_t(C)),\end{aligned}$$

we see that Ξ_t is the composition of the homeomorphisms Υ_t (Lemma 6.3.2) and $\mathcal{K}_t$ (Lemma 6.3.4), so Ξ_t is a homeomorphism.

This completes the proof that Ξ is an isotopy from $(\mathbf{P}, \mathbf{C})$ to $(\mathbb{S}^2_1, \mathcal{C}_c)$.

Chapter 7

Conclusion

In this thesis we have provided a thorough classification of embeddable spherical circle planes by giving explicit descriptions of the topological structures associated with them. The direct methods employed to describe the geometric operations of joining points and intersecting circles, and subsequently the homeomorphic types of circle and flag spaces, have allowed us to prove new results whilst maintaining accessibility to an audience only familiar with elementary topology. Moreover, our constructions have been visualisable in $\mathbb{R}^3$, as demonstrated by the numerous diagrams littered throughout the thesis, so they are didactically pleasing.

Although the problem of classifying the circle space of a general spherical circle plane is still an open one, we have been able to extend the family of spherical circle planes for which the classification is known, giving hope that the axiom of touching may indeed be unnecessary to prove their flag spaces are homeomorphic to the 3-dimensional projective space minus a point.

Having transposed the notion of isotopy equivalence from stable planes to spherical circle planes, we have obtained a new measure to compare spherical circles planes. One may wish to determine isotopy equivalences among different classes of spherical circle planes, and perhaps use it as a further means of classification. One might also wish to emulate our approach and determine how isotopy equivalence can be defined on various other topological geometries. The results of this thesis, then, should not only serve to have extended what we know of spherical circle planes, but to inspire further investigations in the topic.

Appendix

A.1 Results from General Topology

We list certain theorems from general topology used in this thesis. Each is either proven here or includes a direct reference to a proof in the literature. The reader may also use those references to obtain a refresher on the terminology in use; the terms for which we do not provide definitions are those we regard as standard.

A.1.1 Properties of Quotient Spaces

Theorem A.1.1 ([3], Proposition 2.2.2)**.** *Let X, Y be spaces such that $p : X \to Y$ is a quotient map, and let Z be a space. Then a map $f : Y \to Z$ is continuous if and only if the composite map $f \circ p : X \to Z$ is continuous.*

Theorem A.1.2. *Let X, Y, X^* and Y^* be spaces and let $p : X \to X^*$, $q : Y \to Y^*$ be quotient maps. Let $g : X \to Y$ be a continuous map and let $h : X^* \to Y^*$ be a map satisfying the commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ X^* \arrow[r, "h"'] & Y^* \end{tikzcd}$$

Then h is continuous.

Proof. Since $h \circ p = q \circ g$ is continuous, Theorem A.1.1 implies that h is continuous. $\square$

Theorem A.1.3 ([3], Proposition 2.4.15)**.** *Let X, Y be spaces such that $p : X \to Y$ is a quotient map. Then the restriction-corestriction $p|_A : A \to p(A)$ of p to an open (or closed) saturated subspace $A \subseteq X$ is a quotient map .*

Theorem A.1.4 ([3] Theorem 3.3.17)**.** *Let $p : X \to Y$ be a quotient map and let Z be a locally compact space. Then*

$$p \times \mathrm{id}_Z : X \times Z \to Y \times Z$$

is a quotient map.

Theorem A.1.5. *Let $p : A \to B$ and $q : C \to D$ be quotient maps. If B and C are locally compact spaces then $p \times q : A \times C \to B \times D$ is a quotient map.*

Proof. From Theorem A.1.4, the map

$$p \times q : A \times C \xrightarrow{p \times \mathrm{id}_C} B \times C \xrightarrow{\mathrm{id}_B \times q} B \times D$$

is the composition of quotient maps, so is itself a quotient map. $\square$

A.1.2 Miscellaneous Theorems

Theorem A.1.6. *The restriction $f|_A : A \to f(A)$ of an open map $f : X \to Y$ to an open subspace $A \subseteq X$ is an open map.*

Proof. If U is open in the subspace A of X, then $U = U' \cap A$ for some open subset U' of X. But A is open in X and hence so is U. Whence $f(U)$ is open in X, so $f(U) = f(U) \cap f(A)$ is open in the subspace $f(A)$ of Y. $\square$

Theorem A.1.7 ([5], Corollary 8.3)**.** *Open or closed subsets of locally compact, Hausdorff spaces are locally compact.*

Theorem A.1.8 (Gluing lemma; [5], Theorem 7.3)**.** *Let A, B be* closed *subsets of a space X such that $X = A \cup B$; let $f : A \to Y$ and $g : B \to Y$ be continuous mappings. If $f(x) = g(x)$ for all $x \in A \cap B$, then the mapping $h : X \to Y$, defined by*

$$h(x) = \begin{cases} f(x) & x \in A, \\ g(x) & x \in B, \end{cases}$$

is well-defined and continuous.

Theorem A.1.9 ([5], Theorem 5.6)**.** *Let X and Y be topological spaces such that X is compact and Y is Hausdorff. Let $f : X \to Y$ be a continuous, bijective map. Then f is a homeomorphism.*

Theorem A.1.10. *Let (x_n) be a sequence in a compact metric space X such that every convergent subsequence (there is at least one) converges to the point $x \in X$. Then $x_n \to x$.*

Proof. Suppose that $(x_n) \not\to x$; then there is an $\varepsilon > 0$ and a subsequence (x_{n_k}) such that $x_{n_k} \notin \mathcal{B}(x, \varepsilon)$ for all k. By compactness, (x_{n_k}) has a convergent subsequence $(x_{n_{k_l}})$, which in turn is a convergent subsequence of (x_n) and hence must converge to x by the hypothesis. But $x_{n_{k_l}} \notin \mathcal{B}(x, \varepsilon)$ for all l — a contradiction. We deduce that no such subsequence (x_{n_k}) exists, so $(x_n) \to x$. □

Theorem A.1.11 ([12] Theorem 13.7)**.** *A space X is Hausdorff if and only if the "diagonal" set $D := \{(x, x) : x \in X\}$ is closed in the product space $X \times X$.*

Theorem A.1.12 ([12], Theorem 16.2)**.** *The continuous open image of a second countable space is second countable.*

Theorem A.1.13 (Jordan-Brouwer separation theorem; [9], Theorem 4.8.15)**.** *An (n-1)-sphere embedded in* $\mathbb{S}_1^n$ *separates* $\mathbb{S}_1^n$ *into two components of which it is the common boundary.*

Theorem A.1.14 ([2], Proposition 16.3)**.** *Let* C *be a convex, compact subset of* $\mathbb{R}^d$ *(*$d \in \mathbb{R}$*) with* $\mathbf{0} \in \operatorname{Int}(C)$*. Then the map* $f : \partial C \to \mathbb{S}_1^{d-1}$*, given by* $f(\mathbf{x}) = \frac{\mathbf{x}}{||\mathbf{x}||}$*, is a homeomorphism.*

A.2 The Hausdorff-Convergence Topology

To verify that H, as defined in Chapter 5, is indeed a topology on $\mathcal{H}(\mathbb{S}^2)$ we first show that the following familiar property of convergent sequences in a metric space holds for convergent sequences in the Hausdorff-convergence topology.

Proposition A.2.1. *A sequence* (M_n) *in* $\mathcal{H}(\mathbb{S}^2)$ *converges to* $M \in \mathcal{H}(\mathbb{S}^2)$ *with respect to* H *if and only every subsequence* (M_{n_k}) *of* (M_n) *converges to* M *with respect to* H*.*

Proof. First suppose that $(M_n) \xrightarrow{HC} M$ and let (M_{n_k}) be a subsequence of (M_n). It suffices to show that

$$\limsup_k M_{n_k} \subseteq M \subseteq \liminf_k M_{n_k}.$$

• $\limsup_k M_{n_k} \subseteq M$: If $\mathbf{q}$ is an accumulation point of some sequence $(\mathbf{q}_{n_k}) \in (M_{n_k})$, then $\mathbf{q}$ is an accumulation point of $(\mathbf{q}_n) \in (M_n)$, so $\mathbf{q} \in \limsup_n M_n = M$.
• $M \subseteq \liminf_k M_{n_k}$: Let $\mathbf{q} \in M = \liminf_n M_n$. Then, as $\mathbb{R}^2$ is a metric space, we have that

$$\mathbf{q} = \lim_{n\to\infty} \mathbf{q}_n = \lim_{k\to\infty} \mathbf{q}_{n_k}$$

for every subsequence $(\mathbf{q}_{n_k}) \in (M_{n_k})$. Hence $\mathbf{q} \in \liminf_k M_{n_k}$.

Conversely, if every subsequence (M_{n_k}) of (M_n) to M with respect to $\mathcal{H}$, then so does (M_n), being a subsequence of itself. $\square$

Theorem A.2.2. *The Hausdorff-convergence topology* H *is a topology on* $\mathcal{H}(\mathbb{S}^2)$.

Proof. Since $\liminf \varnothing = \limsup \varnothing = \varnothing$, a sequence of empty sets converges in $\mathcal{H}$ to the empty set, so $\varnothing$ is H-closed. Obviously the entire collection $\mathcal{H}(\mathbb{S}^2)$ is H-closed.

We now show that the finite union of H-closed sets is H-closed. By induction it suffices to consider the union of two H-closed subsets $A, B \subseteq \mathcal{H}(\mathbb{S}^2)$. Let (M_n) be a sequence in $A \cup B$ and suppose that $M_n \xrightarrow{HC} M$ for some $M \in \mathcal{H}(\mathbb{S}^2)$. If $M_n \in A$ for only finitely many n then there is an $N > 0$ such that $M_n \in B$ for all $n > N$. Hence, since B is H-closed, we have that

$$M = \lim_{n\to\infty} M_n = \lim_{\substack{n\to\infty \\ n>N}} M_n \in B \subseteq A \cup B.$$

Similarly one finds that if $M_n \in B$ for only finitely many n, then $M \subseteq A \cup B$.

Now suppose there are infinitely many n for which $M_n \in A$ and $M_n \in B$, respectively. Let (M_{n_k}) and (M_{n_l}) be the corresponding subsequences of (M_n) in A and B, respectively. By Proposition A.2.1, we have that $(M_{n_k}) \xrightarrow{HC} M$ and $(M_{n_l}) \xrightarrow{HC} M$. Hence, as A and B are each H-closed we obtain that $M \in A \cap B \subseteq A \cup B$. This shows that $A \cup B$ is H-closed.

Finally we verify that arbitrary intersections of H-closed sets are H-closed. Let $\{A_\alpha\}_\alpha$ be a family of H-closed sets and let (M_n) be a sequence in $\bigcap_\alpha A_\alpha$ such that $(M_n) \xrightarrow{HC} M$ for some $M \in \mathcal{H}(\mathbb{S}^2)$. Then, for each α, we have that (M_n) is a sequence in A_α and $(M_n) \xrightarrow{HC} M$, so $M \in A_\alpha$. Hence $M \in \bigcap_\alpha A_\alpha$, so $\bigcap_\alpha A_\alpha$ is H-closed. $\square$

Proposition A.2.3. *Let* Y *be a metric space and endow* $\mathcal{H}(\mathbb{S}^2)$ *with the Hausdorff-convergence topology. Suppose that* $g : Y \to \mathcal{H}(\mathbb{S}^2)$ *is map such that whenever* *there is a sequence* (y_n) *in* Y *converging to a point* y*, we have* $g(y_n) \xrightarrow{HC} g(y)$*. Then* g *is continuous.*

Proof. Let A be a closed subset of $\mathcal{H}(\mathbb{S}^2)$; we show that $g^{-1}(A)$ is closed in Y. Since Y is a metric space, this is equivalent to showing that $g^{-1}(A)$ contains its limit points in Y. Let (y_n) be a sequence in $g^{-1}(A)$ converging to a point $y \in Y$. Then $g(y_n) \in A$ for each n, and by the hypothesis on g we have that $g(y_n) \xrightarrow{HC} g(y)$. But A is H-closed, so $g(y) \in A$ and hence $y \in g^{-1}(A)$. Whence $g^{-1}(A)$ contains its limit points. $\square$

List of Symbols

Bibliography

[1] R Goldman. "Intersection of Three Planes." *Graphic Gems*, ed. A Glassner, Academic Press, C.A., USA, 1990. 305.

[2] G Bredon. *Topology and Geometry.* Springer-Verlag, New York, 1993.

[3] R Engelking. *General topology.* Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989.

[4] G McCarty. *Topology: An Introduction with Application to Topological Groups.* McGraw-Hill, USA, 1967.

[5] J Munkres. *Topology: a First Course.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1975.

[6] B Polster and G Steinke. *Geometries on Surfaces.* Cambridge University Press, Berlin, 2001.

[7] N Rosehr. A note on the isotopy problem for $\mathbb{R}^2$-planes. *Arch. Math.*, 81:710–720, 2003.

[8] H Salzmann, D Betten, T Grundhöfer, H Hähl, R Löwen, and M Stroppel. *Compact Projective Planes: With an Introduction to Octonion Geometry.* De Gruyter expositions in mathematics, Berlin, 1995.

[9] E Spanier. *Algebraic Topology.* McGraw-Hill, New York-Toronto, 1966.

[10] K Strambach. Sphärische Kreisebenen. *Math. Z.*, 113:266–292, 1970.

[11] K Strambach. Der Kreisraum einer sphärischen Möbiusebene. *Monatsh. Math.*, 78:156–173, 1974.

[12] S Willard. *General topology.* Dover Publications, Mineola, N.Y., 2004.

[13] D Wölk. Topologische Möbiusebenen. *Math. Z.*, 93:311–333, 1966.